Comput Manag Sci
https://doi.org/10.1007/s10287-018-0300-6

ORIGINAL PAPER

On the construction of hourly price forward curves for electricity prices

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Received: 21 August 2017 / Accepted: 7 February 2018

Abstract There are several approaches in the literature for the derivation of price forward curves (PFCs) which distinguish among each other by the procedure employed for the derivation of seasonality shapes, smoothing technique and by the design of the optimization procedure. However, a comparative study to highlight the strengths and weaknesses of different methods is missing. For the construction of PFCs we typically incorporate the information about market expectation from the observed futures prices and the deterministic seasonal effects of electricity prices. In most existing approaches, the seasonality shape is fitted to historically observed spot prices, and it is an exogenous input to the optimization procedure. As seasonal effects on electricity prices differ between markets, our model allows a more general and flexible definition of the seasonality shape. In this study, we propose an alternative calibration procedure for the seasonality shape, where the level of futures as well as historical spot prices are simultaneously taken into account in a joint optimization approach. We discuss

F. Paraschiv: Part of the work has been done during my research time as Assistant Professor at the University of St. Gallen, Insitute for Operations Research and Computational Finance. This research is part of the activities of SCCER CREST, which is financially supported by the Swiss Commission for Technology and Innovation (CTI).

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Published online: 28 February 2018

comparatively the features of existing methods for PFCs, and highlight the advantages of our optimization procedure.

Keywords Electricity markets · Hourly price forward curves · Smoothing techniques · Seasonality shapes

1 Introduction

As most industrial costumers of an utility[1] are heavily dependent on electricity for production purposes and have very little flexibility in demand, they need to minimize the risk induced by highly-volatile electricity prices. Similarly, a producer of electricity will be interested to hedge this risk and thus to secure the level of the price today for the delivery of electricity at a future period of time. This becomes highly relevant, since electricity suppliers must cover their production costs and, in addition, electricity is non-storable and it must be consumed immediately as it is produced. Consumers and producers of electricity will thus insure the (continuous) delivery of electricity over a certain period of time in the future. Futures contracts for electricity are however standardized for delivery of power over a limited set of delivery periods: Over 1 week, 1 month, one quarter or 1 year. There is a limited number of traded Futures contracts at The European Power Exchange (EPEX): weekly, monthly, quarterly and yearly, which restricts the flexibility of market participants to adjust to price levels which typically differ for different hours of 1 day, weekdays and seasons. For this purpose, market participants use the information from price forward curves (PFCs) or hourly price forward curves (HPFCs) to read the fair price for individual days or hours. This becomes highly relevant for example for electricity consumers with specific load profiles, where the consume of electricity is concentrated at specific hours.

Updated PFCs are of particular interest nowadays especially in countries like Germany, where there has been a continuous increase of the infeed of wind and photovoltaic for the electricity production (Erni 2012; Hildmann et al. 2013). Renewable energies are highly volatile and difficult to forecast accurately. Thus, weather updates are observed until short before the delivery period and weather forecasting errors are incorporated in the price formation process in the intraday electricity market (Kiesel and Paraschiv 2017), which implies a high uncertainty around the spot price level. It is therefore relevant to have access to accurate expectations of prices for each hour of the day, which is the goal of hourly price forward curves. The standardization of forward prices along the price forward curve is a hedge against the volatile spot electricity prices and allows market participants to plan better their production and balance out consumption in the future.

For the construction of PFCs we typically incorporate the information about market expectation from the observed futures prices and the deterministic seasonal effects of electricity prices. There are several methods in the literature for the construction of PFCs as shown for example in Fleten and Lemming (2003), Benth et al. (2007), Paraschiv et al. (2015) and Caldana et al. (2017). These differ among each other with

[1] Utility companies are companies such as electric, gas and water firms.

respect to the method chosen for the seasonality shape, to the smoothing component, and with respect to the methodology of getting arbitrage free curves. The typical seasonality patterns of electricity prices contain yearly, weekly and daily patterns which determine ultimately the shape of the demand profile for electricity.

In this study, we firstly discuss the different mathematical models used for the construction of the seasonality shapes. We discuss further the effect of different seasonality shapes on the final resulting PFC. We implemented the existing methods of Fleten and Lemming (2003), Benth et al. (2007), Paraschiv et al. (2015) and discuss comparatively the features of the generated PFCs. On top of this we propose a novel method for the construction of the PFC. The main feature of our model is that we do not treat the seasonality shape exogenously, as it is done in Fleten and Lemming (2003) and Benth et al. (2007), but formulate a more flexible optimization model, where we simultaneously shape and align the curve to the level of the observed Futures prices in a joint optimization procedure. This is insofar important, since it allows a more direct comparison of PFCs in different energy markets with slightly different patterns of the seasonality curves. We will test and compare the selective models with respect to their ability to replicate and forecast the observed electricity prices, which is an additional contribution of this study to the existing literature on PFCs.

The rest of the paper is organized as follows: in Sect. 2 we give a review of the different approaches used for the construction of the PFCs, and we make a comparative assessment of these modeling approaches. In Sect. 4 we compare the different estimated curves with respect to the observed spot prices and Sect. 5 concludes.

2 A review and comparative assessment of modeling approaches for price forward curves

In this section, we give a review and comparative assessment of most popular approaches for the derivation of PFCs. We underline advantages and drawbacks and motivate the need for a new modeling approach. All methods employed in the literature concerning the construction of HPFCs follow the same generic principles. The construction of an HPFC is usually split into three parts.

- First we construct the seasonal curve, which indicates how the prices are distributed throughout the year.
- The second step is to adjust this curve by making it arbitrage free with respect to the observed Futures. We will call this the adjustment part/function of the HPFC.
- As a third step, to get hourly prices, we will need to apply an hourly profile to the daily prices.

2.1 Seasonality component of electricity prices

The seasonality curve is constructed by fitting appropriate periodic functions to historical spot prices. We assume that the typical seasonality patterns are recurrent each year. The seasonality curve will contain yearly, weekly and daily components. The seasonal patterns occur due to weather conditions or economic and business activities.

Yearly Seasonality This is related to natural phenomenons, as different temperatures between summer and winter seasons, which determines a different demand pattern for electricity. The yearly seasonality is also related to vacation and holiday periods where economic activity and thus the use of energy is reduced.

Weekly Seasonality Electricity prices are generally higher during the week, when the economic activity is intense, than during the weekend. Therefore, one typically observes a jump in prices when going from working day to weekend/holiday, therefore we will include dummy variables for the different days of the week, to correct for this pattern.

Daily Seasonality The daily seasonality of electricity prices is determined by the economic activity within 1 day. Typically, one observes lower prices during the night, prices start increasing during the morning hours and reach a peak around noon. It has been empirically observed that the noon peak has flattened over time because of the increasing in-feed of renewable energies (Paraschiv et al. 2016). In winter one typically observes a second evening peak in the German market related to the extra need of heating as people come home from work.

The yearly cycles are typically modeled by trigonometric functions which produce a smooth shape. The other patterns of the seasonality shape related to economic activity (weekly and daily) are typically modeled by dummy variables.

In this study, we will consider three types of seasonality functions, dummy variables, Fourier series or splines. We will compare different methods used in the literature and explain the strengths and weaknesses of the proposed approaches. In particular, we compare the methods from Paraschiv et al. (2015), Fleten and Lemming (2003) and Benth et al. (2007) with a novel model based on trigonometric splines.

2.2 Review of different functions used for the seasonality shape

We give a review of the main functions used for modeling the seasonality patterns. We refer here to both dummy variable related models and trigonometric functions.

Dummy Variables Paraschiv et al. (2015) model the combined yearly and weekly seasonality curve by a mixture of dummy variables and continuous variables for the cooling/heating degree days (CDD/HDD) for three different German cities, defined as follows:

$$s(t) = a_0 + \sum_{i=1}^{6} b_i D_{di} + \sum_{i=1}^{12} c_i M_{di} + \sum_{i=1}^{3} d_i CDD_{di} + \sum_{i=1}^{3} e_i HDD_{di} \tag{1}$$

where a_0 can be interpreted as the mean level of the year. The rest of the terms shape the weekly and the yearly cycles by daily and monthly dummy variables (D_{di}) and (M_{di}) respectively, and it is further stylized by the CDD/HDD.

We empirically observed that the problem with modeling the seasonality curve by dummy variables is that they mainly account for the change in the price level between months, while in reality one expects the price changes to occur more smoothly. Fleten

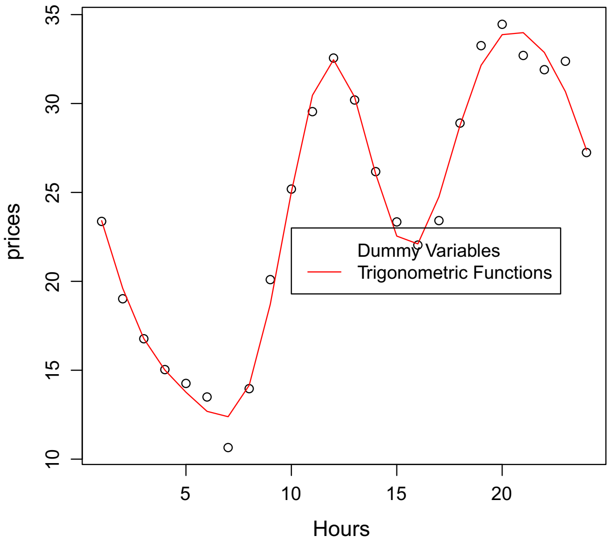

Fig. 1 The circles reflect the observed mean price for each hour of the day during the years 2000–2007 and the continuous line corresponds to the fitted Fourier series of the form $F_4(t) = a_0 + \sum_{i=1}^{4} a_i \sin(2\pi i t/24) + b_i \cos(2\pi i t/24)$ to the same data

and Lemming (2003), which we will describe in Sect. 2.3, cope with this problem by smoothing the HPFC by the adjustment function.

Fourier Series Truncated Fourier series are sums of trigonometric functions:

$$F_n(t) = a_0 + \sum_{i=1}^{n} [a_i \sin(i \cdot \pi \cdot t) + b_i \cos(i \cdot \pi \cdot t)]$$

and are commonly used to model cycles. The reason for this is that they have a natural periodicity, depending on their frequency. The advantage of these functions compared to dummy variables is that they are continuous, meaning there are no sudden jumps between periods. The use of Fourier series for seasonality functions is common, given their simplicity, they are also used for other commodities.[2]

However, the pattern produced by trigonometric functions is too regular, and we cannot model effectively all characteristic price changes. In Figure 1 we show an example of an hourly profile estimated by Fourier series. The fit to the data seems in general good, apart from hours 7 and 23. At hour 7 the mean price is much lower than estimated by the Fourier series, which is probably an effect of the fact that at this hour power plants are turned on to cover the typical increase in the demand during the morning, resulting in an overproduction at that hour, driving the prices down. In a

[2] In a newer study Caldana et al. (2017) propose a method for the construction of HPFCs, using a trigonometric function for the yearly seasonality and dummy variables for the weekly/daily patterns. They model the adjustment function by the monotone convex interpolator (MCI) put forward by Hagan and West (2006).

Table 1 Comparison of the error estimate between a daily profile estimated by truncated Fourier series and dummy variables, column one and three are the errors for the Fourier series, while two and four for Dummy variables, L1 is absolute error, while L2 is square error

	L1		L2	
	Fourier	Dummy	Fourier	Dummy
Hour 1	**10.46224**	10.47887	**172.57410**	173.00676
Hour 2	**10.86177**	11.10551	**220.81164**	226.57419
Hour 3	**13.52108**	13.52546	**252.97063**	253.10604
Hour 4	10.70268	**10.69568**	215.86698	**215.64835**
Hour 5	10.42771	**10.25740**	187.07475	**182.18272**
Hour 6	10.68178	**10.40417**	186.40433	**177.21438**
Hour 7	**10.75718**	11.37051	**247.72809**	263.28876
Hour 8	**10.23698**	10.30623	**174.67778**	176.73565
Hour 9	10.144247	**9.620673**	152.941690	**138.612149**
Hour 10	9.767245	**9.713098**	138.157351	**136.937834**
Hour 11	**9.919567**	10.061363	**145.067145**	146.545121
Hour 12	10.90129	**10.88703**	170.46746	**170.28211**
Hour 13	**11.48788**	11.51683	**194.35153**	194.93814
Hour 14	11.46580	**11.44212**	233.74733	**233.48758**
Hour 15	11.79799	**11.61616**	286.25036	**284.31261**
Hour 16	**11.21104**	11.22297	**253.73688**	253.91762
Hour 17	**10.35865**	10.55824	**180.02174**	184.50132
Hour 18	11.27468	**11.24566**	190.57921	**189.55610**
Hour 19	12.05114	**11.65319**	231.49264	**215.45815**
Hour 20	12.02813	**11.76629**	237.87038	**227.60371**
Hour 21	**10.32580**	10.99499	**175.48254**	195.58561
Hour 22	**9.176717**	9.689951	**139.533732**	151.995862
Hour 23	11.49978	**10.41221**	202.69297	**173.77011**
Hour 24	**9.990992**	10.057976	**163.474317**	165.134128
Mean	10.87718	**10.85844**	198.08231	**197.09979**

The bold faced numbers represent the best fit

similar way, one observes that the prices increase at hour 23, which can be interpreted by the fact that power plants shut down gradually adjusting to off-peak demand.

In the following we test how daily profiles constructed by Fourier series and by dummy variables fit the observed prices during the years 2008–2015. In Table 1 we show the absolute and square differences between the observed and modelled prices, see details in Appendix 5.1

As one can see in Table 1, results are inconclusive, so choosing one method over the other might not matter much for the overall fit. As observed in Fig. 1 there are deviations of the approximation by the Fourier series from the observed mean prices for some specific hours (hour 7 and hour 23 are examples). In such cases, a combined approach of approximation with Fourier and the inclusion of specific dummy variables for hours where deviations occur will be a better choice.

Spline Functions Spline curves are functions $s : [0, T] \to \mathbb{R}$, where we split the time interval $[0, T]$ into smaller intervals $0 = t_1 < t_2 < \cdots < t_{n+1} = T$.We then define basis functions $s_i(t)$

$$s(t) = s_i(t);\, t_i \leq t < t_{i+1},$$

where $s_i(t);\ i = 1 : n$ is a well defined function.[3] We ensure continuity of the curve by setting $s_i(t_{i+1}) = s_{i+1}(t_{i+1})$, and continuous derivatives by setting $s_i^{(n)}(t_{i+1}) = s_{i+1}^{(}n)(t_{i+1})$, where $s_i^{(n)}(t)$ is the n'th derivative of $s_i(t)$. The advantage of the spline over dummy variables is that it generates continuous curves while it still offers greater flexibility than standard trigonometric functions. This type of curve is not commonly used in the literature, but Benth et al. (2007) considers a polynomial spline to adjust the PFC to the observed Futures prices. We will later propose a trigonometric spline, where the basis functions are of the form

$$\begin{aligned} f(t; m) = C + \sum_{i=1}^{6} \left[a_i \sin\left(\frac{2\pi i(t + S(m))}{12 \cdot M(m)}\right) + b_i \cos\left(\frac{2\pi i(t + S(m))}{12 \cdot M(m)}\right)\right] \\ + a_4^{Q(m)} \sin\left(\frac{8\pi(t + S(m))}{12 \cdot M(m)}\right) + b_4^{Q(m)} \cos\left(\frac{8\pi(t + S(m))}{12 \cdot M(m)}\right) \\ + a_{12}^{Q(m)} \sin\left(\frac{24\pi(t + S(m))}{12 \cdot M(m)}\right) + b_{12}^{Q(m)} \cos\left(\frac{24\pi(t + S(m))}{12 \cdot M(m)}\right) \end{aligned}$$

where we allow for different frequency's $\frac{S(m)}{M(m)}$ to cope with the different length of the different months. We also allow for different parameter values for the different quarters, by changing $a_.^{Q(m)}, b_.^{Q(m)}$. We will describe the curve in detail in Sect. 3

Overall one should not use one specific de-seasonalisation approach in isolation, but rather use a combination of the different techniques to reflect observed seasonal patterns in electricity prices.[4]

2.3 Review of existing models

In the current study, we discuss two different popular approaches for the derivation of the price forward curves, namely Fleten and Lemming (2003) and Benth et al. (2007). In these first studies the seasonality shapes have been historically derived and represent an exogenous input for the derivation of the price forward curves. Both optimization procedures have as main objective the minimization of the distance between the seasonality curve and the resulting price forward curve under certain constraints. The curve should be arbitrage free. In Fleten and Lemming (2003) and Benth et al.

[3] $s_i(t) \in C^{\infty}([t_i, t_{i+1}))$.

[4] Finally we should point out that it is common to use other fundamental variables in the construction of the seasonality curves. Standard variables as heating/cooling degree days, demand forecasts, fuel prices or weather forecasts for wind and photovoltaic are also typically sued as additional explanatory variables (Paraschiv et al. 2014). However, this is not the scope of our current study, but it is and interesting subject of future research.

(2007) it is assumed that the PFC $f(t)$ can be decomposed into a seasonal component $s(t)$ and a residual term $\epsilon(t)$ modeling the difference between the seasonality curve and the PFC.

In the sequel we show the mathematical formulation of the two approaches, showing their advantages and their drawbacks. There are two ways of fitting the curve to the futures prices, either when the market is still open, and we observe bid and ask spreads (ex-ante), which is done in Fleten and Lemming (2003), or after the market closes and final prices are observed (ex-post), as done in Benth et al. (2007). Both methods can be adjusted to either of the two approaches. For comparison purposes we will use the ex-post approach in both cases.

In both approaches we let $\Phi = \{(T_1^s, T_1^e), (T_2^s, T_2^e), \ldots, (T_m^s, T_m^e)\}$ be a list of start and end dates for m average-based forward contracts. We collect all starting and end dates in chronological order (overlapping contracts are split in sub-periods). The constructed hourly price forward curve f_t replicates the currently observed market prices $F(T^s, T^e)$ perfectly, where T^s and T^e are the start and end dates for different settlement periods.

Fleten and Lemming (2003) approach Fleten and Lemming (2003) model the hourly price curve by combining the information contained in the observed bid and ask prices with the information about the shape of the seasonal variation.

Let f_t be the price of the forward contract with delivery at time t, where time is measured in hours, and let $F(T_1, T_2)$ be the price of forward contract with delivery in the interval $[T_1, T_2]$. Since only bid/ask prices can be observed, we have:

$$F(T_1, T_2)_{bid} \leq \frac{1}{\sum_{t=T_1}^{T_2} \exp(-rt/a)} \sum_{t=T_1}^{T_2} \exp(-rt/a) f_t \leq F(T_1, T_2)_{ask} \tag{2}$$

where r is the continuously compounded rate for discounting per annum and a is the number of hours per year. When constructing our PFCs, we will assume that the interest rate $r = 0$, in all methods, to have comparable results.

$$\min_{f_t} \left[\sum_{t=1}^{T} (f_t - s_t)^2 + \lambda \sum_{t=2}^{T-1} (f_{t-1} - 2f_t + f_{t+1})^2 \right] \tag{3}$$

The parameter lambda controls for the smoothness of the curves: $\lambda = 0$ means no smoothing, and if $\lambda \rightarrow \infty$ the originally forecasted seasonality shape will be obsolete, meaning that if one constructs a PFC from two different seasonality shapes, the resulting curves will converge to the same when $\lambda \rightarrow \infty$. In Fig. 2 we have constructed four PFCs, one with a real seasonality curve and one where the data is drawn from a normally distributed random variable. One can observe that when λ is small the resulting PFCs differ a lot, while when λ is big they are quite similar in both cases.

In Fig. 3 we show the difference between the PFC and the seasonality shape, where we applied the Fleten and Lemming (2003) approach. In the original model of Fleten and Lemming (2003), applied for daily steps, a smoothing factor prevents large jumps in the forward curve. However, in the case of an hourly resolution of the curves

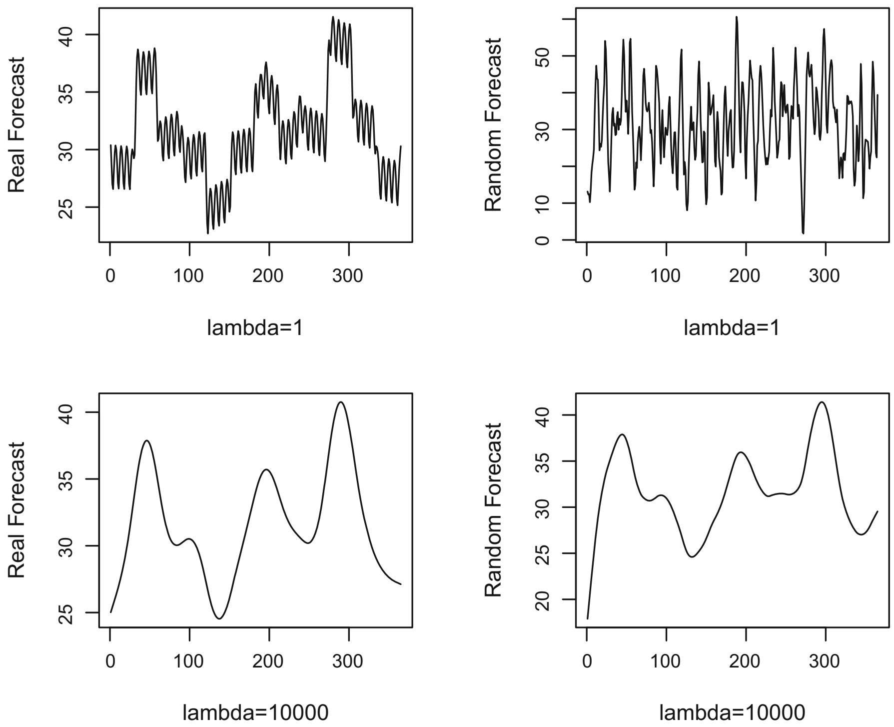

Fig. 2 Showing four different PFCs constructed by the method described in Fleten and Lemming (2003), two curves made with a forecast from dummy variables and two curves where the forecast is just drawn from a normal random variable, with $\lambda = 1$ and $\lambda = 10{,}000$

(HPFCs), Blöchlinger (2008, p. 154), concludes that the higher the relative weight of the smoothing term. In Fig. 4 we see how the daily seasonality is suppressed in the method by Fleten compared to the two other methods in question.

Benth et al. (2007) approach In the method suggested by Benth et al. (2007) the constructed hourly price forward curve $f(t)$ replicates the currently observed market prices $F(T_s, T_e)$ perfectly, where T_s and T_e are the start and end dates for different settlement periods:

$$F(T^s, T^e) = \frac{1}{T^e - T^s} \int_{T^s}^{T^e} f(t)\, dt, \tag{4}$$

where $f(t)$ consists of a seasonality curve $s(t)$ and a correction term $\varepsilon(t)$. The correction term $\varepsilon(t)$ is modeled by a polynomial spline of the form:

$$\varepsilon_t = \begin{cases} a_1 t^4 + b_1 t^3 + c_1 t^2 + d_1 t + e_1 & t \in [t_0, t_1) \\ a_2 t^4 + b_2 t^3 + c_2 t^2 + d_2 t + e_2 & t \in [t_1, t_2) \\ \vdots & \\ a_n t^4 + b_n t^3 + c_n t^2 + d_n t + e_n & t \in [t_{n-1}, t_n] \end{cases} \tag{5}$$

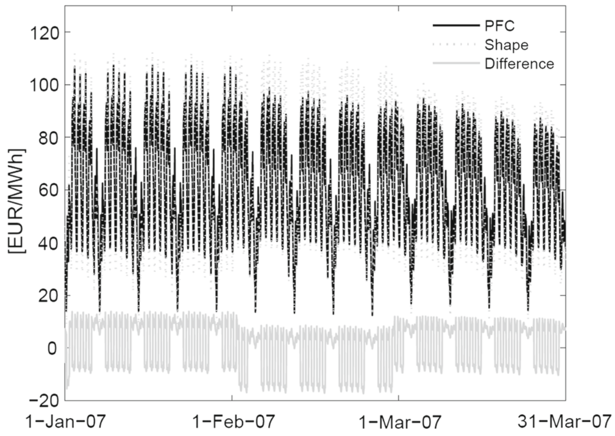

Fig. 3 Shape–HPFC Q1 2007. *Source*: Blöchlinger (2008)

$$x^T = [a_1\ b_1\ c_1\ d_1\ e_1 \quad a_2\ b_2\ c_2\ d_2\ e_2 \ldots a_n\ b_n\ c_n\ d_n\ e_n] \tag{6}$$

The minimization criterion that will ensure our curve has maximum smoothness is given by:

$$\min_x \int_{t_0}^{t_n} [\varepsilon''(t; x)]^2 dt \tag{7}$$

To ensure continuity and continuous derivatives throughout the periods further conditions are required (see Appendix 5.2).

2.4 Critical view

One of the differences in the two approaches is that the smoothing of the curve is done in different ways. As we have seen, in the Fleten and Lemming (2003) approach the smoothing is done directly on the curve, while in Benth et al. (2007) the smoothing is done on the correction term by splines.

The approach in Fleten and Lemming (2003) The problem with the first approach is twofold: Firstly, the lambda parameter of the smoothing factor has an aleatory nature, there is no common agreement in the literature about its size. Secondly, since the smoothing is done directly on the curve, it suppresses the daily and weekly patterns of the seasonality shape (see Blöchlinger 2008). This can be a serious drawback when one is interested in PFCs of higher resolution.

A solution to the fact that the smoothing suppresses the weekly and hourly pattern can be to reapply these patterns after the smoothing is done. In this way we can first ensure a smooth curve, and afterwards ensure that we have a sufficient daily/weekly seasonality. We will later show evidence that such an approach gives better results when looking at the weekly seasonality.

As for the aleatory nature of the parameter λ one solution could be to choose the smallest λ that results in a smooth enough curve. The meaning of "smooth enough"

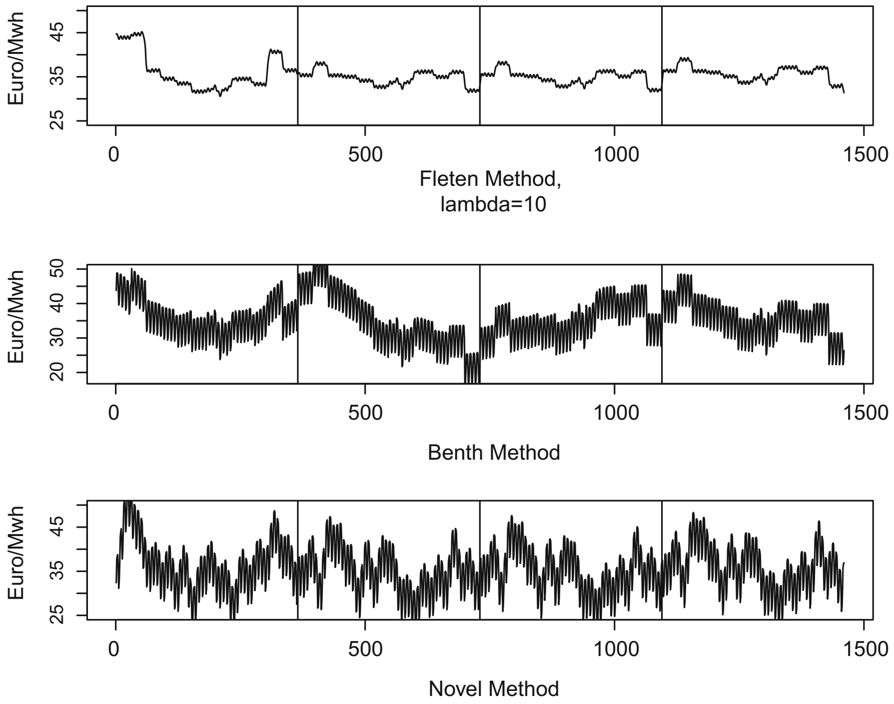

Fig. 4 The three PFCs estimated for a 4 year period (2018–2021), with data input observed 7th of December 2017. The black vertical lines represent the start/end of each year

is that the largest price difference between two consecutive days is less than a pre-set number and not too high compared to the difference between other consecutive days.

The approach in Benth et al. (2007) In the approach by Benth et al. (2007) the weekly and daily seasonality, as the seasonality patterns are not affected by the smoothing. As a result, if one uses a non-smooth seasonality curve, as is the case with dummy variables, the result will be a non-smooth HPFC.

In Fig. 4 we show the three curves with a 4 year granularity, generated at 07/12/2017. We observe that the patterns are recurring for the method proposed by Fleten, and our novel method. While in the method by Benth we observe alternating upward/downward trends for the last 3 years, where only one yearly Futures product is used in the calibration.

A drawback with the approach by Benth et al. (2007) is that the number of parameters used in the fitting of the curve is dependent on the number of Futures observed. In Fig. 5 we have constructed two adjustment curves, one where the three first monthly Futures and three quarterly Futures are used as input. For the derivation of the second curve we took also took the price coming from the PFC for the 4th month as input in the construction of the PFC, leaving the other Futures prices used as input for the first curve unchanged. The result is shown in Fig. 5. We observe that even if our original PFC correctly estimates the price of the 4th monthly Futures price, using this price as input in the calibration will change the curve. This means that once a new maturity is

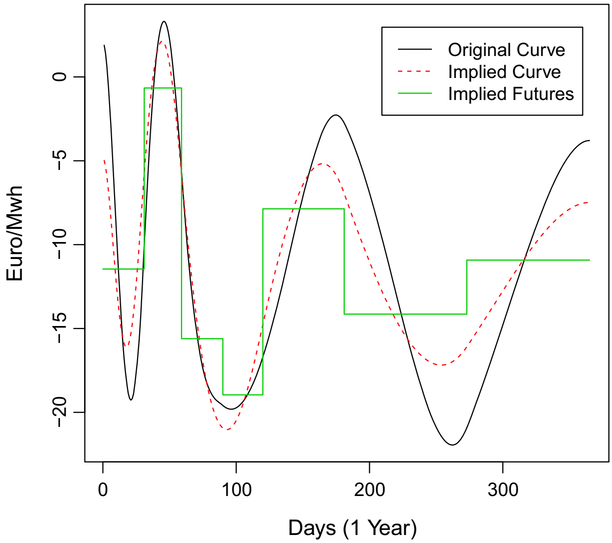

Fig. 5 The adjustment functions generated based on the approach by Benth et al. (2007), the black line is generated on the following input: the observed, de-seasonalized, prices for the first three monthly Futures and three quarterly Futures; the second curve has as additional input the 4th month Future. The straight lines are the corresponding Futures prices the second curve is fitted to

becoming available in the market, this will change the PFC in a deterministic manner. By adding Futures of other maturities will alterate the market expectation for all forward prices along the curve. This result shows a shortcoming of the method of Benth et al. (2007). This result is meaningful in particular when new maturities becomes available from 1 day to the other, which will lead to two different curves.
In both approaches, the seasonality shape is calibrated to historical spot prices, and it is exogenously inserted in the optimization problem. That means that the forecasted shape replicates the historical oscillations in prices. However, it has been empirically observed that the increasing infeed from wind and photovoltaic in Germany has decreased the level of electricity prices over time (see Paraschiv et al. 2014, 2016). In consequence, the traditional spreads between peak and off-peak power prices have been narrowed. Furthermore, due to the volatile renewable energies, in particular in case of very cold or hot years, the seasonality shape can no longer be considered as standard and it is more difficult to construct it accurately. In this context, the reliance on pure historical prices for the derivation of the seasonality shape cannot realistically reflect the current dynamics.

To overcome this methodological drawback, we propose a novel approach for the derivation of PFC's, where we allow the seasonality shape to reflect historical oscillations, but at the same time we adjust the amplitude to the observed Futures prices.

3 Novel modeling approach for PFCs

In the novel modeling approach we propose a joint optimization procedure where the seasonality shape is not treated exogenously, but it is simultaneously fitted to the historical spot prices and to the currently observed Futures prices. We believe that the amplitude of the oscillations along the seasonality curve should fit the market expectation about the level of the Futures prices with different delivery periods.

Mathematical specification of the novel model

We model the seasonality curve and the correction term by one trigonometric spline, which is defined as follows:

$$f(t;m) = C + \sum_{i=1}^{6}\left[a_i \sin\left(\frac{2\pi i(t+S(m))}{12\cdot M(m)}\right) + b_i \cos\left(\frac{2\pi i(t+S(m))}{12\cdot M(m)}\right)\right] \quad (*)$$

$$+ a_4^{Q(m)} \sin\left(\frac{8\pi(t+S(m))}{12\cdot M(m)}\right) + b_4^{Q(m)} \cos\left(\frac{8\pi(t+S(m))}{12\cdot M(m)}\right) \quad (**)$$

$$+ a_{12}^{Q(m)} \sin\left(\frac{24\pi(t+S(m))}{12\cdot M(m)}\right) + b_{12}^{Q(m)} \cos\left(\frac{24\pi(t+S(m))}{12\cdot M(m)}\right) \quad (***)$$

The spline part is divided into two parts: For each month m, we have a different seasonality, as a multiple of $12\cdot M(m)$ where

$$M(m) = \{\#\ \text{ days in month } m\}$$

instead of 365. We chose this to have a better correspondence between the length of the different months and the PFC. We also allow for more variability for each quarter, by having different parameters $a_{\cdot}^{Q(m)}, b_{\cdot}^{Q(m)}$ for each quarter. The parameter $S(m)$ is chosen to insure continuity of the curve. For details, see 5.3

Here t is the time in days parameter, $1 \leq t \leq 365$, and m is a counting parameter for the months while $M(m)$ is the corresponding number of days in that month:

Explanation of the different terms The different parts of the function generating the PFC can be explained in this way:

The first part (*) will not differ significantly from a standard truncated Fourier series, but this choice of periodicity links the PFC to the months, and therefore to the Futures prices better.

As a standard truncated Fourier series can be too regular to correctly estimate the complex structure of electricity prices, we will add more flexibility by including a spline trigonometric curve, by the terms with the superscript $Q(m)$ in lines denoted by (**)and (***). These parameters are allowed to vary across quarters. The terms in line (**) will account for the flexibility of the curve, while the terms in line (***), ensure continuity and continuous derivatives. The fact that we have different parameters in the different quarters represents the spline part of the curve.

The constant C represents the mean level of the curve, while the other parts will describe how the prices distribute throughout the year. From now on we will refer to the first term colored in black as the Fourier term, and the parts in red and blue as the spline terms.

Parameter selection

The choice of the number of parameters in the Fourier term was determined by using Lasso regression trying to determine the number of significant factors. We started with 24 different terms and reduced it to 12, but there is still reason to believe that the number of relevant factors can be improved, especially by also changing the number of spline terms. This leads to the following set of 26 parameters:

$$x = (a_1, \ldots, a_6, b_1, \ldots, b_6, a_4^2, \ldots, a_4^4, b_4^2, \ldots, b_4^4, a_{12}^1, \ldots, a_{12}^4, b_{12}^1, \ldots, b_{12}^4)$$

The terms a_4^1 and b_4^1 cancel against a_4 and b_4

Fitting of the curve

The general idea behind the fitting procedure is: if a class of functions shows a reasonable fit to observed historical seasonalities, then these functions should also be able to replicate simultaneously the observed prices of traded Futures products. In our model, we reflect the seasonality pattern of spot prices by the trigonometric functions introduced before and simultaneously align the seasonality curve to the observed Futures. Since our seasonality curve is linear in the parameters, this is the same as solving a constrained least squares optimization problem.

Our problem reads as follows:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \|Ax - y\|_2 \\ \text{subject to} \quad & Cx = V \end{aligned}$$

where Ax (see Appendix 5.3 for the specification of A) is our seasonality linear function and y represents the historical spot prices. In the constraints matrix C, we will ensure the no arbitrage condition by imposing that the PFC correctly replicates the observed Futures prices. As we are working with a trigonometric spline, the matrix C also needs to include the continuity constraints. A solution is obtained by solving the linear problem:

$$\begin{bmatrix} 2A^T \cdot A & C^T \\ C & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 2A^T \cdot y \\ V \end{bmatrix}$$

If the matrix on the left-hand side is invertible, the optimal solution $\hat{x}$ is defined by:

$$\begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 2A^T \cdot A & C^T \\ C & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2A^T \cdot y \\ V \end{bmatrix}$$

It should be noted that if the solution x^{OLS} from the ordinary least squares problem, obtained by fitting the model to only the historical spot prices, already solves $C\hat{x}^{OLS} = V$, then these two solutions coincide.

The matrix C together with the vector V corresponds to the constraints and can be decomposed into two matrices:

$$C = \begin{pmatrix} H \\ G \end{pmatrix}$$

where H corresponds to the Futures and G corresponds to the constraints needed on the spline part of the curve. The solution to our optimization problem x^* gives us the desired price forward curve $f(t)$, which is computed by the matrix multiplication Ax^*. The curve here does not include a weekly or daily seasonality, and is therefore meant to describe the distribution of the prices throughout the year. The weekly and daily seasonalities can be included by methods described earlier in this paper.
This approach depends on the fact that we use a seasonality function that is linear in the parameters. However, it is flexible enough that one can use the same method by taking a seasonality curve based on the standard Fourier series, dummy variables or some other class of functions that are linear in the parameters.

Downsides with the novel modeling approach As argued for earlier, while the evolution between normal days is expected to be smooth, typically when going to and from holiday periods jumps are observed and these characteristics are hard to model with a smooth curve.

The model introduced above also does not include a term designed for taking care of the weekly seasonality, so this curve represents how the prices are distributed throughout the year, excluding the weekly pattern. In our estimation results we will use a weekly seasonality component modeled by dummy variables, as in Paraschiv et al. (2015). One can either add the dummy variables directly in the optimization method, or one can add a weekly seasonality after the optimization.

In Fig. 6 we show our PFC as well as how a typical adjustment curve, modeling the difference between the PFC and the seasonality curve will look like in this approach.

4 Estimation results

In this section, we will asses comparatively the performance of the various methods discussed in this study to generate PFCs. We assume that every forward price of a certain maturity along the PFCs should meet in expectation the realized spot price. We are aware that there are deviations between the price forward curve and the realized spot prices due to the risk premium component. However, an estimation of the risk premium is not in the scope of this study. We assume the risk premium is the same for all price forward curves and thus a comparison for PFCs is realistic.

4.1 Data used

We have generated PFCs based on four different methods. To test the validity of the curves, we constructed two sets of HPFCs, each including three curves, based on the methods discussed in this study: Fleten and Lemming (2003), Benth et al. (2007) and

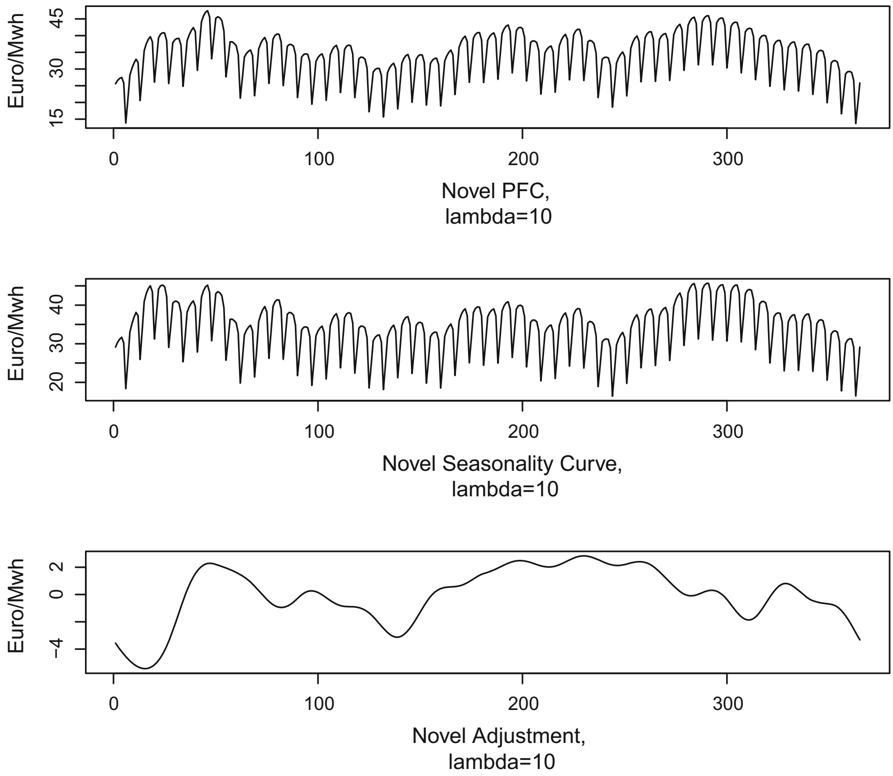

Fig. 6 In panel 1 we show our PFC for 2015 made of a trigonometric spline to model the yearly seasonality and dummy variables to model the weekly seasonality. The Futures products used as input are six monthly variables and two quarterly variables. In panel 2 we show the same PFC, but with no Futures prices used as input as a proxy for the seasonality curve. In panel 3 we show the difference between these curves, to show how we fit our PFC to the Futures products

our novel approach. One fourth curve was generated based on Fleten and Lemming (2003) where we added ex-post the daily and weekly seasonal pattern from Eq. (1). The reason is that the standard approach of Fleten and Lemming (2003) suppresses the weekly and daily seasonal patterns if we include the smoothness. All curves are generated for the year 2015.

The first set of curves are estimated based on historical spot prices from 2011 to 2013 used to fit the seasonality curve and on Futures products observed in 2014 for the three first months and the remaining three quarters in 2015. This will be our out-of-sample analysis. The second set of curves will be our benchmark, they were constructed by taking the observed spot prices for 2015 and as a proxy for the Futures we took average of the realized spot prices over the different months for the corresponding delivery period. This will be our in-sample analysis. For the methods by Fleten and Lemming (2003) and Benth et al. (2007) we will use a seasonality curve based on dummy variables, as described in Paraschiv et al. (2015). Due to the technical specification of our model we cannot take the same dummy based seasonality shape in the comparative assessment of the produced HPFCs.

Table 2 Show the absolute mean of estimated week mean–real week mean for week 1 to 52, as calculated from in-sample data from 2015

Week	Novel method	Fleten	Benth	Week	Novel method	Fleten	Benth
1	**1.34**	5.78	5.89	27	**1.43**	1.85	2.32
2	**5.52**	8.41	8.45	28	1.34	**0.00**	0.79
3	**3.69**	7.19	7.18	29	**4.30**	5.53	5.56
4	**0.98**	6.18	6.13	30	**3.37**	5.33	4.31
5	4.63	**3.75**	3.92	31	**3.53**	4.37	3.66
6	**0.65**	2.17	1.93	32	**0.22**	2.66	3.21
7	**1.80**	1.84	2.02	33	**0.15**	3.67	3.80
8	3.68	2.20	**2.05**	34	**1.95**	3.77	4.66
9	**3.00**	5.37	5.25	35	8.35	1.33	**0.22**
10	5.16	**3.02**	3.11	36	**0.69**	3.78	4.06
11	**1.33**	2.24	2.15	37	1.05	**0.38**	0.41
12	**3.15**	5.38	5.05	38	1.66	**1.31**	1.51
13	**7.19**	7.48	7.86	39	1.66	**1.49**	2.32
14	1.77	**1.24**	1.79	40	**1.53**	3.31	3.96
15	**2.73**	3.61	3.32	41	1.95	**1.34**	1.35
16	**0.69**	1.73	1.76	42	**2.96**	4.25	4.51
17	**0.94**	1.70	1.48	43	1.70	1.58	**1.00**
18	**0.45**	0.80	0.84	44	**3.67**	5.07	4.96
19	**0.27**	1.95	2.09	45	2.15	**0.80**	1.31
20	1.20	**0.13**	0.56	46	**3.69**	4.50	4.72
21	**1.78**	2.60	1.88	47	4.50	**2.81**	2.84
22	**1.18**	1.86	2.20	48	1.40	**1.13**	1.26
23	1.90	1.77	**1.44**	49	**1.38**	4.33	4.17
24	2.42	**0.92**	1.08	50	**3.33**	5.30	5.36
25	2.55	**0.17**	0.23	51	**2.62**	3.05	2.83
26	2.12	**1.76**	1.80	52	**3.59**	7.10	6.75

The bold faced numbers represent the best fit

4.2 Comparative assessment of generated price forward curves

The set of curves have been generated for a weekly daily and hourly resolution and then compared to average observed weekly, daily and hourly spot prices.

In Tables 2 and 3 we show for each method the in- and out of sample performance as:

$$|Estimated\,Price\,Week_w - Real\,Price\,Week_w|$$

where $Estimated\,Price\,Week_w$ is the generated price from the PFCs, and $Real\,Price\,Week_w$ is the observed average spot price for the corresponding week.
As seen in Table 2, the novel modeling approach scores best for 33 out of 52 weeks, while the other methods score best for 15 and 4 weeks, respectively. This comes from

Table 3 Show the absolute mean of estimated week mean–real week mean for week 1 to 52, as calculated from out-of-sample data

Week	Novel method	Fleten	Benth	Week	Novel method	Fleten	Benth
1	17.69	8.25	**4.79**	27	18.98	**15.97**	19.31
2	7.23	6.29	**4.88**	28	19.49	**16.63**	19.50
3	**4.79**	9.32	10.46	29	14.84	**12.64**	14.66
4	**11.08**	15.63	18.59	30	**9.14**	9.25	12.34
5	**8.97**	11.69	12.96	31	14.14	**14.09**	16.31
6	**0.67**	3.83	3.67	32	8.64	**8.12**	9.68
7	**8.56**	10.61	10.86	33	15.34	**14.91**	15.37
8	18.15	**14.24**	14.96	34	19.52	19.33	**17.69**
9	23.30	**18.62**	19.28	35	15.88	17.10	**14.97**
10	17.45	**17.03**	17.33	36	13.24	15.10	**12.36**
11	**17.82**	19.96	20.01	37	12.57	14.53	**11.43**
12	**17.07**	18.18	17.71	38	16.73	18.99	**15.53**
13	21.05	20.15	**18.90**	39	16.12	18.32	**14.84**
14	19.39	20.20	**17.66**	40	**10.69**	11.42	12.03
15	17.04	18.43	**15.72**	41	**12.23**	13.91	14.54
16	21.72	21.89	**19.22**	42	16.65	**14.98**	15.33
17	14.94	13.54	**10.98**	43	20.67	**14.71**	14.78
18	20.62	20.32	**19.05**	44	17.45	13.74	**12.94**
19	**25.91**	27.12	26.33	45	**10.37**	14.51	13.00
20	**25.63**	26.69	26.39	46	**11.73**	18.92	17.12
21	25.07	**24.40**	24.65	47	**17.14**	19.55	17.56
22	23.25	**22.20**	23.52	48	13.31	13.35	**12.15**
23	**25.57**	25.93	28.50	49	**6.19**	10.36	11.84
24	**24.73**	25.37	28.36	50	**8.00**	11.60	12.91
25	14.98	**14.38**	17.59	51	11.94	**9.01**	10.20
26	20.39	**18.45**	21.83	52	6.79	1.79	**0.50**

The bold faced numbers represent the best fit

the fact that the methods by Fleten and Lemming (2003) and Benth et al. (2007) generate relatively flat curves during 1 month, as observed in Fig. 7 (the price for the estimated weekly forward prices are constant within 1 month), while the novel approach allows for more variability during the course of 1 month. This comes from the fact that the novel modeling approach uses continuous functions as a basis for the seasonality curve instead of dummy variables. Thus, our novel approach is more parameter intensive, which helps to shape better the curve. However, this feature might lead to over-fitting, which can explain why our model performs better in-sample, but it loses accuracy out of sample as we observe in Table 3. In general, when we go out of sample we observe an overall increase in the deviations between the observed average weekly spot prices and the estimated prices for all models. The increase in the

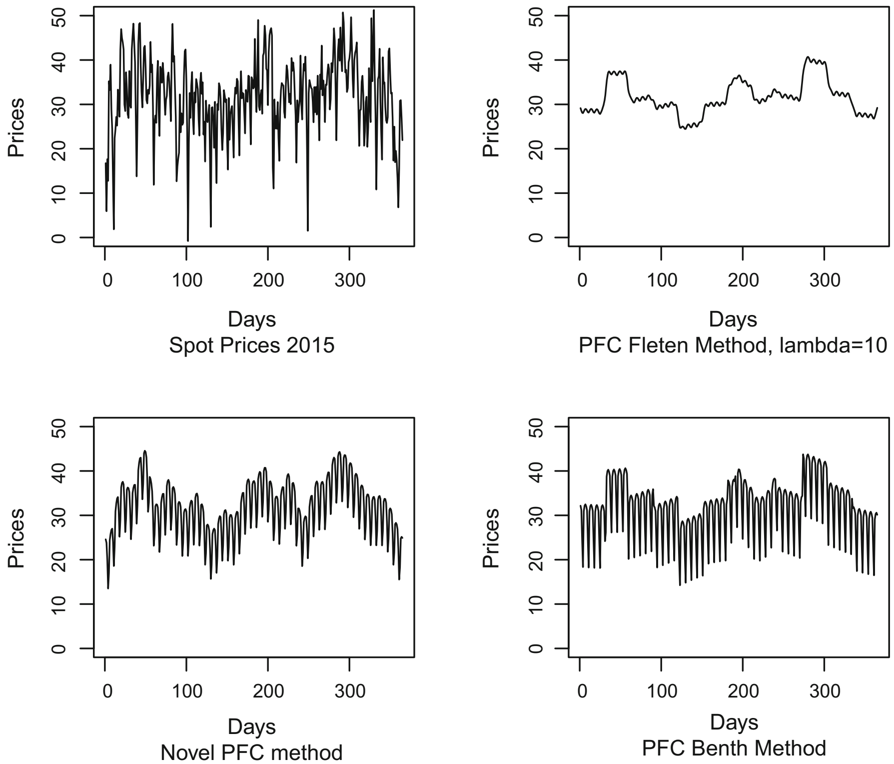

Fig. 7 The graph in the top left panel show the evolution of the spot prices used for the in sample calibration. The three other graphs represents the PFCs generated based on the three different methodologies

errors in the out of sample case study shows that historical data are not a good enough estimator of the future market expectations of electricity prices.

We compare further the performance of the four PFCs in fitting the realized prices based on the following statistics: We computed the absolute, the squared error and the mean average percentage error (MAPE). Results are available in Table 4.

$$Absolute\,Error = \frac{1}{n}\sum_{i=1}^{n} |Realized\,Price_i - Estimated\,Price_i| \tag{8}$$

$$Squared\,Error = \frac{1}{n}\sum_{i=1}^{n} (Realized\,Price_i - Estimated\,Price_i)^2 \tag{9}$$

$$MAPE = \frac{100\%}{n}\sum_{i=1}^{n} \frac{|Realized\,Price_i - Estimated\,Price_i|}{|Realized\ Price_i|} \tag{10}$$

The novel modeling approach scores best for all the in sample tests, while the second method of Fleten and Lemming (2003) scores best for the out of sample tests. This result can be related to the differences in the technical specifications of the models: In

Table 4 Comparison of the different models to the realized spot prices

Daily Scale		Benth	Novel	Fleten 1	Fleten 2
Test	Data				
MAPE	In sample	32%	**29%**	45%	32%
MAPE	Out of sample	67%	42%	57%	**41%**
Absolute difference	In sample	4.79	**4.57**	6.05	4.85
Absolute difference	Out of sample	8.76	6.79	7.15	**6.32**
Hourly Scale		Benth	Novel	Fleten 1	Fleten 2
Test	Data				
Absolute difference	In sample	5.95	**5.83**	7.15	5.98
Absolute difference	Out of sample	9.92	8.09	8.55	**7.67**
Square difference	In sample	65.71	**61.69**	91.46	65.89
Square difference	Out of sample	181.69	116.86	139.76	**109.25**

Fleten 1 is the original Fleten method, while Fleten 2 is where we have reapplied the weekly seasonality. The bold faced numbers represent the best fit

our approach the over-fitting property of the seasonality function applied to historical prices leads to larger out of sample errors. In Fleten and Lemming (2003) where we used the exogenous defined seasonality shape based on dummy variables we get a more rough approximation of the historical seasonality, which leads to a slightly better out of sample fit. In any case, there are no major differences between the various methods in the in- and out of sample results.

5 Conclusion

In this study, we compare different methods proposed in the literature for the derivation of hourly price forward curves and discuss the effects of various input components on the resulting curves: seasonality functions, adjustment (smoothing approaches) as well as the overall optimization procedures. In particular, we discuss the effect of adjustment functions proposed in Fleten and Lemming (2003) and Benth et al. (2007) combined with the seasonality function based on dummy variables from Paraschiv et al. (2015). In addition to the comparative view, we formulate a novel approach for the derivation of forward curves where we perform a combined fitting of the seasonality curve and the adjustment function based on trigonometric splines.

Concerning the various types of seasonality curves employed in the literature as input to the derivation of electricity price forward curves, our results conclude that: A seasonality function based on *dummy variables* does not allow for a continuous curve, resulting in large price jumps between periods, typically months, which is an undesirable feature of the HPFC. A functional form of the seasonality curve as the *trigonometric spline* leads instead to the opposite problem. It is not able to model sudden price movements which typically occur in the electricity price behaviour when moving from week to weekend days or between individual hours, when new power plants are taken in/out of the production-mix to cope with peak/off-peak hours.

In the method proposed by Fleten and Lemming (2003), the smoothing is done directly on the seasonality curve while aligning the curve to the observed Futures prices which suppresses the hourly and daily seasonality patterns. In our tests, this method performs best out-of-sample if one exogenously reshapes the weekly/daily seasonality pattern. This result is due to the flexible formulation of the optimization procedure which avoids overfitting and can be more generally applied.

The method by Benth et al. (2007) uses a polynomial spline of the fourth degree to model the adjustment function. We have discussed two weaknesses of this method: One is that the smoothing is only done on the adjustment function, which is positive if one is already satisfied with the smoothness of the seasonality curve (exogenously defined), but not suitable if one wants to smooth parts of the seasonality curve. The other weakness is that the number of parameters are dependent on the number of Futures products observed, resulting in a deterministic change of the curve when new products are added to the market, which can be used to form arbitrage strategies.

Our novel method is based on a constrained least square optimization procedure where the underlying function is a trigonometric spline. We observe that this method is the best for replicating the spot prices in an in-sample test, but does not outperform the other models in the out-of-sample tests. We attribute this to two reasons: In this framework we need more variables for the seasonality curve. When we do not observe all Futures maturities typically available in the market we obtain more free variables, leading to an over-fitting of the curve. Secondly, this method allows for more variability within 1 month resulting in different prices for the different weeks in 1 month, which is not the case for a curve based on dummy variables. Thus, including too many variables for the seasonality curve might lead to over-fitting. Overall our method is innovative as it proposes a joint optimization procedure where the seasonality curve, smoothness and the alignment of the curve to the observed Futures are integrated in a joint optimization procedure.

Our novel method clearly outperforms the others, in its ability to estimate the weekly seasonality, as keeping the price relatively constant during 1 month is too restrictive. A seasonality curve offering more flexibility will outperform a dummy variable curve, if one can correctly estimate the intra-month seasonalities, but our out-of-sample analysis show a pure reliance on historical spot prices might be too restrictive for this. As for our approach of fitting the seasonality curve directly to the Futures prices is preferable if one believes that the seasonality curve should be able to replicate these Futures prices, but should be used in combination with another adjustment function to replicate products with short delivery periods. This to keep the number of parameters low, and reduce the risk of over-fitting. The method in Fleten and Lemming (2003) would be fitting in combination, as it can alter the price of individual days, without affecting the curve as a whole. The results shown in this paper show that a combination of different methods is preferable, over using only one method for the seasonality curve and adjustment function, dependent on the specific need. Our model has the advantage over the two other models that it does not suppress the seasonality, as in Fleten and Lemming (2003), and it does not produce deterministic jumps in the PFC when the granularity of the observed Futures products change, as in Benth et al. (2007).

Appendix

5.1 Testing Procedure Table 1

The tests for Table 1 is done as follows: We obtain our estimated prices by taking the real price for each day d multiplied by the hourly profiles to get an estimated price for each hour h in day d. Then we take the mean of the absolute differences between this estimate and the observed price for hour h at day d for all days in 2008–2015. The same is done for the squared differences.

$$\text{Absolute\_Error\_Hour}_h = \frac{1}{n}\sum_{d=1}^{n} |Day\,Price_d \cdot Hourly\,Profile_h - Hour\,Price_{d,h}| \tag{11}$$

$$\text{Squared\_Error\_Hour}_h = \frac{1}{n}\sum_{d=1}^{n} (Day\,Price_d \cdot Hourly\,Profile_h - Hour\,Price_{d,h})^2 \tag{12}$$

5.2 Constraints in the method by Benth et al. (2007)

Throughout the periods, the following Eqs. (13)–(15) need to hold:

$$(a_{j+1} - a_j)t_j^4 + (b_{j+1} - b_j)t_j^3 + (c_{j+1} - c_j)t_j^2 + (d_{j+1} - d_j)t_j + e_{j+1} - e_j = 0 \tag{13}$$

$$4(a_{j+1} - a_j)t_j^3 + 3(b_{j+1} - b_j)t_j^2 + 2(c_{j+1} - c_j)t_j + d_{j+1} - d_j = 0 \tag{14}$$

$$12(a_{j+1} - a_j)t_j^2 + 6(b_{j+1} - b_j)t_j + 2(c_{j+1} - c_j) = 0 \tag{15}$$

To ensure that the curve is flat in the long end, we set the first derivative in the end point equal to 0, ensured by Eq. (12). To account for settlement of the contracts throughout the period, one can include a function $w(r;t)$ as shown in Eq. (13). For settlement only at the end points, one sets $w(r;t) = 1/(T_i^e - T_i^s)$.

$$\varepsilon'(t_n; x) = 0 \tag{16}$$

$$F_i^C = \int_{T_i^s}^{T_i^e} w(r;t)(\varepsilon(t) + s(t))dt \tag{17}$$

5.3 Novel modeling approach

The term $S(m)$ is chosen to insure continuity of the curve between the transition times of the months. As an example, when going from January to February, we get $M(1) = 31$ and $M(2) = 28$, where the transition between January and February takes place when $t = 31$. This means that for the curve to be continuous, we need that:

$$\frac{31 + S(1)}{31} = \frac{31 + S(2)}{28}$$

The correction term $S(\cdot)$ that ensures the continuity is given by the vector defined as:

$$S = (0, -3, 3, 0, 4, -1, 5, 5, -3, 6, -4, 7)$$

where $S(m)$ is element number m in the vector S.

The matrices A needed for the optimization in the novel modeling approach are defined as follows:

$$A = \left[A_1, \ldots, A_6, B_1, \ldots, B_6, A_4^1, \ldots, A_4^4, B_4^1, \ldots, B_4^4\right] \tag{18}$$

where:

$$A_j = \left(\sin\left(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}\right), \ldots, \sin\left(\frac{2\pi \cdot j \cdot 31}{12 \cdot 31}\right), \sin\left(\frac{2\pi \cdot j \cdot 25}{12 \cdot 28}\right), \ldots, \sin\left(\frac{2\pi \cdot j \cdot 341}{12 \cdot 31}\right)\right)$$
$$B_j = \left(\cos\left(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}\right), \ldots, \cos\left(\frac{2\pi \cdot j \cdot 31}{12 \cdot 31}\right), \cos\left(\frac{2\pi \cdot j \cdot 25}{12 \cdot 28}\right), \ldots, \cos\left(\frac{2\pi \cdot j \cdot 341}{12 \cdot 31}\right)\right)$$

and A_j^Q for $1 \leq Q \leq 4$ is the vector A_j in quarter Q, and 0 otherwise, giving:

$$A_j^1 = \left(\sin\left(\frac{2\pi \cdot j \cdot 1}{12 \cdot 31}\right), \ldots, \sin\left(\frac{2\pi \cdot j \cdot 93}{12 \cdot 31}\right), 0, \ldots\ldots, 0\right)$$

and the same for B_j^Q.

The matrix for the constraints, named C is as said separated into two different matrices, H, G, where H makes sure the PFC fits to the Futures, and G takes care of the continuity of the spline part of the PFC. The constraints coming from the Futures are given as:

$$V_i = \frac{1}{T_E - T_S} \int_{T_S}^{T_E} f(t)dt \tag{19}$$

where we divide by the length of the period the Futures product is covering since the Futures price is denoted by the average price for that period. Assuming we have a Futures product covering January with price V_1, the corresponding constraint for term j of the Fourier series is:

$$\begin{aligned} V_1 &= \frac{1}{31} a_j \int_0^{31} \sin\left(\frac{2\pi \cdot j \cdot t}{12 \cdot 31}\right) + b_j \cos\left(\frac{2\pi \cdot j \cdot t}{12 \cdot 31}\right) dt \\ &= \frac{31 \cdot 12}{31 \cdot 2\pi \cdot j} \left[-a_j \left(\cos\left(\frac{2\pi \cdot j}{12}\right) - \cos(0)\right) + b_j \left(\sin\left(\frac{2\pi \cdot j}{12}\right) - \sin(0)\right)\right] \end{aligned}$$

Which shows two of the advantages with changing the seasonality corresponding to the length of the months: Firstly, one can cancel the terms coming from the dividing by the length of the period directly against the term coming from multiplying with the denominator in the sin / cos term. Secondly, one only need to evaluate sin / cos in values that are a multiple of $2\pi/12$, where one has nice analytic values for the result. One gets similar results if the Futures products covers more than 1 month. By denoting:

$$F(i, j)_C = \frac{12}{2\pi \cdot j} \cos\left(\frac{2\pi \cdot j \cdot (i+1)}{12}\right) - \cos\left(\frac{2\pi \cdot j \cdot (i)}{12}\right)\Bigg) \tag{20}$$

and similar for the sin function by $F(i, j)_S$. Then the matrix C, if all monthly Futures are given, is defined by:

$$H = \begin{bmatrix} F(1,1)_C & F(1,2)_C & \dots & F(1,6)_C & F(1,1)_S & \dots \\ F(2,1)_C & \dots & \dots & \dots & F(2,1)_S & \dots \\ \dots & \dots & & & & \\ F(12,1)_C & \dots & & & & \end{bmatrix}$$

Where the pattern is similar as for the matrix A with the spline coefficients. For the matrix G that ensures continuity and continuity of the derivatives of the splines, one has these constraints:

Continuity:

$$\begin{aligned} & b_4^1 + b_{12}^1 \\ = \; & b_4^2 + b_{12}^2 \\ & b_4^2 + b_{12}^2 \\ = \; & b_4^3 + b_{12}^3 \\ & b_4^3 + b_{12}^3 \\ = \; & b_4^4 + b_{12}^4 \end{aligned}$$

differentiability:

$$\begin{aligned} & \frac{1}{31}\left(a_4^1 + 3a_{12}^1\right) \\ = \; & \frac{1}{30}\left(a_4^2 + 3a_{12}^2\right) \\ & \frac{1}{30}\left(a_4^2 + 3a_{12}^2\right) \\ = \; & \frac{1}{31}\left(a_4^3 + 3a_{12}^3\right) \\ & \frac{1}{30}\left(a_4^3 + 3a_{12}^3\right) \\ = \; & \frac{1}{31}\left(a_4^4 + 3a_{12}^4\right) \end{aligned}$$

Giving us 6 constraints for the 26 parameters, making G a 6×26 matrix.

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