AMBIGUITY AVERSION AND THE EQUITY PREMIUM PUZZLE: THEORETICAL BACKGROUND AND EMPIRICAL EVIDENCE

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ABSTRACT

This thesis provides a theoretical introduction to the Equity Premium Puzzle within a consumption-based asset pricing framework drawing on the idea of a lower bound on the variance of a pricing kernel as introduced by Hansen and Jagannathan (1991). I discuss the asset pricing implications of three different economies with a representative investor exhibiting a. time-separable CRRA preferences, b. risk-sensitive, recursive Epstein-Zin preferences and c. ambiguity-sensitive Hansen-Sargent preferences. Finally, the thesis conducts an empirical analysis of ambiguity aversion, or fear of model-misspecification, on country-level. I follow Barillas, Hansen and Sargent (2009) and calculate maximum implied detection error probabilities for a set of major economies for the period from 1970 – 2011 and – on a sliding window basis – for the U.S. for the periods from 1889 – 2012 and 1947 – 2012, respectively. I show that ambiguity averse preferences as specified by Hansen and Sargent (2001) resolve the Equity Premium Puzzle introduced by Mehra and Prescott (1985) for all economies and time periods under consideration based on passably low levels of ambiguity aversion. Additionally, I find that under certain assumptions about a representative investor’s time preferences inflation may be a major source of ambiguity aversion and that ambiguity aversion explains a significant part of future long-term aggregate fluctuations in real equity prices. I investigate cross-country variations in implied ambiguity aversion and find some evidence that these variations cannot be explained by structural differences between countries.
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1. **Introduction**

The Equity Premium Puzzle is one of the best known and most comprehensively researched issues in the asset pricing literature. The puzzle was introduced by Mehra and Prescott (1985) who find that it is not possible to simultaneously explain the large historical average equity returns and the small average risk-free rate by means of a general equilibrium model: For the period from 1889–1978 they observe an annualised return on U.S. equities, as represented by the S&P 500, around 7% and an annualised risk-free rate, as represented by t-bills, around 1%, resulting in a historical equity premium of roughly 6% per annum. However, a simple pure exchange economy as specified in their paper will not imply average annual excess returns above 0.4% unless implausibly high levels of risk aversion are tolerated. As a by-product of their work, Hansen and Jagannathan (1991) validate these results within a consumption-based asset pricing framework in an elegant and economically intuitive manner: They analytically derive a lower bound on a stochastic discount factor’s variance and show that the associated permissible area can only be reached if the representative investor’s risk aversion coefficient is massively higher than supported by a large number of experimental studies and common sense. Similar to Mehra and Prescott, Hansen and Jagannathan use a very simple specification of the economy. In particular, they assume time-additive, constant relative risk aversion preferences for their representative investor. Tallarini (2000) overcomes this simplification by applying recursive (non-expected) Epstein and Zin (1989) preferences and finds that the risk aversion implied by this kind of model is smaller, but, as many critics claim, still too high. More recently, Barillas, Hansen and Sargent (2009) extend the analysis of Tallarini to the case of ambiguity averse investors. They find that ambiguity aversion has the same effect on expected value and variance of a stochastic discount factor as risk aversion (in a recursive utility framework) and that the Hansen-Jagannathan Bounds are reached — i.e. that the Equity Premium Puzzle is resolved — for reasonably low levels of ambiguity aversion.

This thesis draws on the results of Barillas et al. (2009), replicates their model and subsequently extends their empirical work conducting an analysis of ambiguity aversion, or fear of model-misspecification, on country-level for some of the most important equity markets worldwide. I make the strong assumption that an ambiguity averse specification of the representative investor’s intertemporal preferences leads to a complete resolution of the Equity Premium Puzzle. Based on this assumption, I estimate and compare the implied ambiguity aversion, as measured by detection error probabilities, for different countries for the 1970 – 2011 period. Additionally, I analyse the trailing implied ambiguity aversion for the U.S. stock market based on two different datasets for the periods from 1889 – 2012 and 1947 – 2012.

The rest of the thesis is organised as follows: **Section 2** derives the asset pricing implications of an extremely simple economy with standard time-separable, constant relative risk aversion preferences. This economy serves as a reference model and as a mean to introduce and illustrate the most important concepts and ideas used in subsequent sections. **Section 3** dismisses the idea of time-separable utility: Introducing Epstein-Zin recursive preferences leads to a generalisation of the model in **Section 2**. In **Section 4**, the Hansen-Jagannathan
Bound is introduced and combined with the results from Sections 2 and 3 in order to illustrate the Equity Premium Puzzle within a consumption-based asset pricing framework. In Section 5, ambiguity is defined and the notion of ambiguity aversion as fear of model-misspecification is motivated. The consideration of ambiguity averse preferences and the corresponding asset pricing implications complete the theoretical part of the thesis. Empirical results from the cross-country and the U.S. time-series analyses are presented in Section 6. Section 7 concludes.
2. **Economy I: Time-separable CRRA preferences**

2.1 **The economy**

I consider a simple two-period general equilibrium model as in Cochrane (2005, pp. 3-7). The economy is defined as follows:

There is a utility maximising representative investor with time-separable intertemporal utility of the form:

\[ U(c_t, c_{t+1}) = u(c_t) + \beta E_t(u(c_{t+1})), \tag{2.1} \]

where \( c_t \) is the investor’s consumption in \( t \), \( \beta \) is a subjective discount factor, \( u(\cdot) \) is an (atemporal) utility function used to evaluate the utility associated with a consumption stream \( c \) and \( E_t(\cdot) \) is the expectation conditional on the information set available in \( t, \mathcal{F}_t \).

For the utility function \( u(\cdot) \) I choose the power utility representation:

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{2.2} \]

where \( \gamma \) is a risk aversion coefficient, or, more specifically in this case, the investor’s constant relative risk aversion (CRRA)\(^1\).

Furthermore, there is one asset available at price \( p_t \) in \( t \) which provides a random payoff \( x_{t+1} \) in \( t+1 \). The investor is allowed to acquire a discretionary amount (in units) \( \alpha \) of this asset. Additionally, the investor gets endowments \( e_t \) in \( t \) and \( e_{t+1} \) in \( t+1 \), representing e.g. his labour income or public subsidies.

The investor’s set of feasible consumption plans is then given by the following intuitive budget constraints:

\[ c_t = e_t - \alpha p_t, \tag{2.3a} \]
\[ c_{t+1} = e_{t+1} + \alpha x_{t+1} \tag{2.3b} \]

These budget constraints force the investor to do a trade-off between consumption today and consumption tomorrow: in the first period (2.3a), he gets an endowment and has to decide how to split it between consumption and an investment in the single asset of this economy. In the second period (2.3b), he gets another endowment and additionally the return on the number of assets acquired in the first period. I.e.

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\(^1\) Power utility is used throughout this paper in order to be able to replicate the models of Tallarini (2000) and Barillas et al. (2009). Also, beyond this necessity, it is economically reasonable to choose power utility due to the aforementioned fact that it exhibits CRRA, which is in many cases a desirable property of an investor’s utility specification. As an extra benefit, power utility has the advantage that it is easy to handle mathematically.
extensive consumption in the first period implies a smaller investment and therefore modest consumption in the second period (and vice versa).

2.2 **Asset Pricing Implications**

The investor seeks to maximise his intertemporal utility in 2.1. Hence, the maximisation problem for this economy is given by:

$$\max_{c_t} \ u(c_t) + \beta E_t(u(c_{t+1}))$$

s.t.  
$$c_t = e_t - \alpha p_t$$

$$c_{t+1} = e_{t+1} + \alpha x_{t+1}$$

Substituting $c_t$ and $c_{t+1}$ by the respective constraints 2.3a and 2.3b in the objective function, the problem can be reformulated:

$$\max_{e_t} \ u(e_t - \alpha p_t) + \beta E_t(u(e_{t+1} + \alpha x_{t+1}))$$

Using the chain rule, the first order condition of this problem is given by:

$$-p_t u'(e_t - \alpha p_t) + \beta E_t(x_{t+1} u'(e_{t+1} + \alpha x_{t+1})) = -p_t u'(c_t) + \beta E_t(x_{t+1} u'(c_{t+1})) = 0,$$  \hspace{1cm} 2.4

where $u'(\cdot)$ is a partial derivative with respect to $\alpha$.

Obviously, the investor determines the amount $\alpha$ such that his overall intertemporal utility is maximised.

Solving the first order condition 2.4 for the asset’s price yields\(^2\):

$$p_t u'(c_t) = \beta E_t(x_{t+1} u'(c_{t+1}))$$

$$\Rightarrow p_t = E_t \left( x_{t+1} \beta \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) \right)$$

This relation is known as the fundamental asset pricing equation. It shows that the price of any asset in the market is the expected value of its random payoff times the intertemporal marginal rate of substitution (IMRS) – in this case $\beta \left( \frac{u'(c_{t+1})}{u'(c_t)} \right)$.

\(^2\) Note that $u'(c_t)$ is $\mathcal{F}_t$-measurable and $\beta$ is constant. Thus, it is equivalent to write either $\frac{1}{u'(c_t)} \beta E_t(x_{t+1} u'(c_{t+1}))$ or $E_t \left( x_{t+1} \beta \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) \right)$. 

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Hence, in general the fundamental asset pricing equation can be written in the following form:

\[ p_t = E_t(x_{t+1}m_{t+1}) \]  \hspace{1cm} 2.5

\[ m_{t+1} := \frac{\partial U_t(c_t, c_{t+1})}{\partial U_t(c_t)} = IMRS \]  \hspace{1cm} 2.6

Where \( U_t(\cdot) \) is the intertemporal utility in \( t \), \( \partial \) is the partial derivative symbol and \( m_{t,t+1} \) denotes a pricing kernel or stochastic discount factor (SDF) applicable to the period from \( t \) until \( t+1 \). I will use the notations \( m_{t,t+1} \) and \( m_{t+1} \) interchangeably.

As mentioned before, for Economy I the SDF is given by:

\[ m_{t+1} = \beta \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) \]  \hspace{1cm} 2.7

In the particular case of power utility we have (2.2):

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma} \]

And since \( u'(c) = \frac{(1-\gamma)c^{-\gamma}}{1-\gamma} = c^{-\gamma} \), it holds that:

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \]  \hspace{1cm} 2.8

It is for the sake of economic intuition that I derive the SDF for Economy I in the form above. In the general case – i.e. independent of the investor’s preference specification – the SDF is given by 2.6.

2.3 The SDF: Basic Insights

In order to proceed with the derivation of the models used in this paper and particularly with the introduction of the so called Hansen-Jagannathan Bound in Section 4, it is expedient to stop at this point for a moment and recall some basic and commonly known characteristics of an SDF (see e.g. Cochrane (2005)).

Let \((\Omega, \mathcal{F}, P, F)\) be a filtered probability space where \(\Omega\) is the state space, \(\mathcal{F}\) is a sigma-algebra on \(\Omega\), \(P\) is a physical probability measure on \(\mathcal{F}\) and \(F\) is a filtration. Then \(F_t \in F\) is the information structure available in \(t\). The number of events in \(t+1\), i.e. the cardinality of \(\mathcal{F}_{t+1}\), is denoted by \(\Phi_{t+1}\). Furthermore, I assume a risk neutral probability measure \(Q\), where \(P\) and \(Q\) are equivalent martingale measures. In this case an SDF has the following characteristics:
• An SDF is an $\mathcal{F}$-measurable random variable or a mapping $SDF: \mathcal{F} \to \mathbb{R}_+^\infty$. In other words, an SDF assigns a value from the set $[0, \infty)$ to every element of the sigma algebra $\mathcal{F}$ (an event). In vector-notation, it can therefore be represented by a vector (of length $\phi_{t+1}$) of non-negative elements. The non-negativity constraint on a valid SDF is imposed by the economic intuition of an investor’s non-satiation; marginal utility, and accordingly the IMRS, are assumed to be strictly positive.

• An SDF can be considered as the real world counterpart to the deterministic discount factor used under the risk-neutral measure $Q$. In a risk neutral world, the price of an asset in $t$, $p_t$, is given by:

$$p_t = \frac{1}{R_{t+1}^{\ell}} E_t(x_{t+1}) = E_t \left( x_{t+1} \frac{1}{R_{t+1}^{\ell}} \right)^3,$$

where $x_{t+1}$ is the asset’s random payoff in $t+1$, $R_{t+1}^{\ell}$ is the gross risk-free rate applicable to the respective period and $E_t(\cdot)$ is the expectation operator under measure $Q$ and conditional on time $t$ information, $\mathcal{F}_t$. In a risk neutral world an investor is unconcerned about the risk associated with different realisations of $x_{t+1}$ and thus is allowed to discount any future payoff by the same factor: the inverse of the gross risk-free rate. The effect of the investor’s risk aversion on asset prices is incorporated into the Radon-Nikodym derivative which transforms the physical measure $P$ into the risk neutral measure $Q$.

• In contrast, the SDF allows to price assets under the physical measure as in 2.5:

$$p_t = E_t(x_{t+1}m_{t+1}),$$

where $E_t(\cdot)$ is the expectation under measure $P$. Hence, in this case the investor’s risk aversion is directly accounted for by the SDF. To see why, recall that $m_{t+1} = \beta \left( \frac{u'(c_{t+1})}{u'(c_t)} \right)$ for the standard CRRA case with time-additive, intertemporal preferences. Due to the fact that $c_t$ is $\mathcal{F}_t$-measurable and due to the assumed risk aversion or decreasing marginal utility, the IMRS, and accordingly the SDF, is large for bad consumption states and small for good consumption states in $t+1$, because an additional unit of consumption provides more marginal utility in a state where the investor suffers from poverty (a bad consumption state) than in a state where the investor lives in opulence (a good consumption state).

• It follows that a payoff which occurs in a good consumption state is less worth to an investor than the same payoff which occurs in a bad consumption state. Accordingly, an asset which tends to provide large payoffs in states where an investor is well off anyway and small or negative payoffs in states where the investor is worse off, will have a low price (and vice versa); an investor gets a premium for bearing the risk associated with the cyclicality of such an asset.

\[3\] Note that equality between the two r.h.s.-terms holds due to the fact that the theoretical risk-free rate $R_{t+1}^{\ell}$ is deterministic.


3. **Economy II: Epstein-Zin recursive preferences**

3.1 **The economy**

The main problem with time-additive utility functions is that they imply a strong and frequently undesired inverse relationship between relative risk aversion (RRA) and intertemporal elasticity of substitution (IES)\(^4\). A proof for the power-utility case considered here is given in Appendix 8.1\(^5\).

This inverse relationship is unpleasant because it has a non-intuitive implication: RRA determines an investor’s tendency to smooth consumption across different states, whereas IES determines his tendency to smooth consumption over time. Commonly, one would probably expect a positive relation between these two parameters.

Epstein and Zin (1989) develop a class of recursive utility functions (henceforth EZ-preferences) which overcome this well-known shortcoming of time-separable expected utility preferences. They achieve a separation between risk preferences and intertemporal substitution preferences by introducing an intertemporal substitution parameter additional to the risk aversion parameter commonly known from the expected utility framework. Economy II is characterised by a representative investor who tries to maximise these recursive intertemporal preferences by choosing optimal consumption in every period.

EZ-preferences are given by (Gourio, 2011, p. 36):

\[
\begin{align*}
U_t &= F(c_t, G(U_{t+1})) & \text{(3.1a)} \\
G(U_{t+1}) &= H^{-1}(E_t H(U_{t+1})) & \text{(3.1b)}
\end{align*}
\]

Where \(U_t\) is the investor’s intertemporal utility in \(t\), \(c_t\) the investor’s consumption in \(t\), \(F(\cdot)\) an aggregator function, \(G(\cdot)\) a certainty equivalence function and \(H(\cdot)\) the investor’s (atemporal) utility function used to evaluate continuation values of intertemporal utility.

3.1a can be motivated by the intuition that intertemporal utility today \(U_t\) should be determined by consumption today \(c_t\) in combination with a measure of future intertemporal utility \(G(U_{t+1})\). The aggregator function \(F(\cdot)\) determines how today’s consumption and the measure of future utility are combined.

\(G(U_{t+1})\) is, as mentioned above, the investor’s certainty equivalent with respect to the uncertain future intertemporal utility stream \(U_{t+1}\). This fact becomes evident if the structure of 3.1b is examined\(^6\): The term \((E_t H(U_{t+1}))\) represents expected (atemporal) utility associated with the continuation value of intertemporal utility.

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\(^4\) See e.g. Hall (1985)

\(^5\) Also, from the Appendix it becomes apparent that power utility belongs to the class of CRRA utility functions with RRA equal to the risk aversion parameter \(\gamma\).

\(^6\) Note that \(H(\cdot)\) maps consumption to utility, whereas \(H^{-1}(\cdot)\) maps utility to consumption.
utility. The inverse function \( H^{-1}(\cdot) \) then transforms this measure of utility into a consumption equivalent measure.

Based on these considerations, it can be seen that risk aversion preferences and intertemporal substitution preferences are separated through the nested structure of 3.1a where risk aversion is captured by \( G(\cdot) \) and intertemporal substitution by \( F(\cdot) \).

The functional forms of \( F(\cdot) \) and \( H(\cdot) \) will be the same throughout the thesis, following most of the relevant literature on the topic:\(^7\)

\[
H(U_{t+1}) = \frac{U_{t+1}^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1 \tag{3.2a}
\]

\[
H(U_{t+1}) = \log(U_{t+1}), \quad \gamma = 1 \tag{3.2b}
\]

\[
F(c_t, G(U_{t+1})) = \left( (1 - \beta)c_t^\rho + \beta G(U_{t+1})^\rho \right)^\frac{1}{\beta}, \quad \rho \neq 1 \tag{3.3a}
\]

\[
F(c_t, G(U_{t+1})) = c_t^{1-\beta} + G(U_{t+1})^\beta, \quad \rho = 1 \tag{3.3b}
\]

Or equivalently in log-form:

\[
\log \left( F(c_t, G(U_{t+1})) \right) = (1 - \beta) \log(c_t) + \beta \log(G(U_{t+1})), \quad \rho = 1 \tag{3.3c}
\]

Where \( \gamma \) is a parameter of risk aversion, \( \rho \) a parameter of intertemporal substitution and \( \beta \) a subjective discount factor.

3.2a is the standard formulation of power utility. 3.2b is a special case of power utility for \( \gamma = 1 \), called log-utility. 3.3a and 3.3b are constant elasticity of substitution (CES) aggregators; in the specific case, this means that the elasticity of substitution between consumption today \( c_t \) and the certainty equivalent of the risky future utility stream is constant. The relation between IES and the intertemporal substitution parameter \( \rho \) in 3.3a, 3.3b and 3.3c is given by the inverse relation \( IES = 1/\rho \) (Epstein & Zin, 1991, p. 266).

The functional form of \( G(\cdot) \) is implied by 3.1b and 3.2a (see Appendix 8.2 for the proof):

\[
G(U_{t+1}) = E_t \left( U_{t+1}^{1-\gamma} \right)^\frac{1}{1-\gamma} \tag{3.4^8}
\]

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\(^7\) See e.g. Epstein and Zin (1989) / (1991) and specifically the implicit specification of the type I agent in Barillas et al. (2009).

\(^8\) This is the general specification of the certainty equivalent for the class of CRRA utility functions (see Epstein and Zin (1991, p. 266)).
Putting 3.4 into 3.3a, 3.3b and 3.3c provides complete functional forms for intertemporal utility:

\[ U_t = \left( (1 - \beta)c_t^\rho + \beta \left( E_t \left( U_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{\rho}} \right)^{\frac{1}{\rho}} \]  

3.5a

\[ U_t = c_t^{1-\beta} + \left( E_t \left( U_{t+1}^{1-\gamma} \right) \right)^\beta \]  

3.5b

\[ \log(U_t) = (1 - \beta) \log(c_t) + \beta \log \left( E_t \left( U_{t+1}^{1-\gamma} \right) \right) \]  

3.5c

Tallarini (2000, p. 509) assumes an IES equal to one for his risk-sensitive agent.\(^9\) Due to the inverse relation between IES and the parameter of intertemporal substitution \( \rho \), this implies \( \rho = 1 \). I.e. 3.3b and 3.3c are aggregator functions for CES equal to one. Indeed, the final functional form of the log-aggregator (3.5c) is a version of the intertemporal utility formulation Tallarini uses to characterise his representative investor. In order to obtain exactly the same representation as Tallarini (2000, p. 510), 3.5c can be slightly transformed and rescaled by \((1 - \beta)\) to state:

\[ \frac{\log(U_t)}{(1-\beta)} = \log(c_t) + \beta \frac{1}{(1-\beta)(1-\gamma)} \log \left( E_t \left( U_{t+1}^{1-\gamma} \right) \right) \]  

3.5d

Setting \( V_t = \frac{\log(U_t)}{(1-\beta)} \), \( U_t = e^{(1-\beta)V_t} \) respectively, in 3.5d:

\[ V_t = \log(c_t) + \beta \frac{1}{(1-\beta)(1-\gamma)} \log \left( E_t \left( e^{((1-\beta)(1-\gamma)V_{t+1})} \right) \right) \]  

3.5e

3.5e represents the recursive intertemporal preferences in Economy II and it will be used as a counterpart to the time-additive intertemporal preferences 2.1 in Economy I.

### 3.2 Asset pricing implications

Tallarini (2000, p. 511) finds that the SDF for Economy II is given by (see Appendix 8.3 for the proof):

\[ m_{t+1} = \beta \frac{c_t}{E_t \left( e^{((1-\beta)(1-\gamma)V_{t+1})} \right)} \]  

3.6

---

\(^9\) Usually, IES is fixed at a level which is assumed to comply well with historical consumption data. The assumption that the empirical IES lies around 1 is supported by a number of papers on the topic (see e.g. Beaudry and van Wincoop (1996)).
The term $\beta \frac{V_t}{V_{t+1}}$ is the SDF for Economy I when $\gamma = 1$ (see 2.8). The martingale increment $\frac{e^{(1-\beta)(1-\gamma)V_{t+1}}}{E_t[e^{(1-\beta)(1-\gamma)V_{t+1}}]}$ serves as an adjustment factor to this “standard” SDF: If the investor’s relative risk aversion $\gamma$ is larger (smaller) than unity, payoffs which occur in a state characterised by a positive intertemporal utility shock – i.e. when $V_{t+1}$ is unusually large – are more heavily (moderately) discounted as compared to the standard SDF. Ceteris paribus, $\gamma$ determines the convexity of the SDF in $V_{t+1}$: The larger $\gamma$, the more convex the SDF as a function of the continuation value $V_{t+1}$.

In an economy with EZ-preferences, the risk premium and therefore the price of an asset is determined by the correlation of its payoff with both consumption and (recursive) intertemporal utility.

Another related finding is that, when $\gamma = 1$, the EZ-preferences collapse into the standard CRRA time-separable preferences with log-utility and $IES = 1$; 3.6 is identical to 2.8 in this case (Tallarini, 2000, p. 510).

The above shows that EZ-preferences can be considered as a generalisation of the standard expected utility preferences in Economy I. This generalisation allows keeping the IES constant at a discretionary level.
4. The consumption-based Equity Premium Puzzle

Hansen and Jagannathan (1991) introduce a novel consumption-based asset pricing consideration of the Equity Premium Puzzle. They derive a restricted admissible region for mean / variance pairs of an SDF ($\{\mu(m), \sigma^2(m)\}$-pairs). The lower bound of this admissible region in the ($\mu(m), \sigma^2(m)$)-space is called the Hansen-Jagannathan Bound (HJB). In order that a general equilibrium model is consistent with empirical equity market data, mean and variance of the SDF must lie within this bound. They use the fact that both the mean and the variance of the SDF can be represented as a function of the investor’s risk aversion coefficient $\gamma$. Accordingly, they calculate the minimum value for the risk aversion coefficient $\gamma$ from the set of all risk aversion coefficients which push the $\{\mu(m), \sigma^2(m)\}$-pair into the HJB and they find the same result as Mehra and Prescott (1985), namely that the large empirically observed equity premium could only be justified by an implausibly high risk aversion of investors.

In the following I derive the HJB and the economic intuition behind it. All relevant ideas and results are borrowed from Cochrane’s (2005) standard work on consumption-based asset pricing. In a second step, I apply the HJB to both economies from the previous sections in order to illustrate the Equity Premium Puzzle.

4.1 The Hansen-Jagannathan Bound

4.1.1 Pricing of gross returns

The one-period gross return of a security is defined as its payoff in $t+1$ divided by its price in $t$:

$$R_{t+1} := \frac{x_{t+1}}{p_t}$$

The price of the random variable $R_{t+1}$ can be derived from the fundamental asset pricing equation 2.5 by dividing it by $p_t$:

$$p_t = E_t (x_{t+1} m_{t+1})$$

$$\Rightarrow 1 = E_t \left( \frac{x_{t+1}}{p_t} m_{t+1} \right) = E_t (R_{t+1} m_{t+1}) \quad 4.1$$

Once again, I make use of the fact that $p_t$ is $F_t$-measurable. Relation 4.1 shows that the price of any gross return as defined above is one.

Let $R_{t+1}^f$ be the gross risk-free rate, i.e. the gross return on an asset with deterministic payoff $x_{t+1}^f$. In this particular case $R_{t+1}^f$ is deterministic and the r.h.s.-term in 4.1 can be slightly rearranged to state the following:
Which implies that:

\[ E_t(m_{t+1}) = \frac{1}{R^f_{t+1}} \quad \text{Or equivalently} \quad R^f_{t+1} = \frac{1}{E_t(m_{t+1})} \quad 4.2 \]

This reveals an intuitive relation between the stochastic discount factor and the deterministic discount factor applicable in a risk neutral world.

### 4.1.2 The Risk Premium

The covariance between the SDF and an asset’s payoff is given by:

\[ \text{Cov}_t(m_{t+1}, x_{t+1}) = E_t(m_{t+1}x_{t+1}) - E_t(m_{t+1})E_t(x_{t+1}) \]

Using this relation and 4.2, the fundamental asset pricing equation 2.5 can be reformulated:

\[
P_t = E_t(m_{t+1})E_t(x_{t+1}) + \text{Cov}_t(m_{t+1}, x_{t+1}) = \frac{1}{R^f_{t+1}} E_t(x_{t+1}) + \text{Cov}_t(m_{t+1}, x_{t+1}) \quad 4.3
\]

Hence, the price of an asset can be decomposed into the expected value of its payoff discounted by the risk-free rate – the asset’s quasi-present value if it is pretended to be risk-free – and a risk adjustment determined by the covariance between the asset’s payoff and the SDF.

Using 4.3 and the fact that gross returns have a price of one (4.1) the following representation for the price of gross returns can be derived:

\[ 1 = \frac{1}{R^f_{t+1}} E_t(R_{t+1}) + \text{Cov}_t(m_{t+1}, R_{t+1}) \]

Rearranging yields:

\[ E_t(R_{t+1}) = R^f_{t+1} - R^f_{t+1}\text{Cov}_t(m_{t+1}, R_{t+1}) \quad 4.4 \]

Relation 4.4 states that the expected value of a gross return is equal to the gross risk-free rate plus an asset-specific risk premium. Note that the risk premium \((-R^f_{t+1}\text{Cov}_t(m_{t+1}, R_{t+1})\) is positive for negative (positive) correlation between the gross return and the SDF (consumption) (see also Section 2.3). Again, this implies a positive risk premium associated with cyclical securities.
4.1.3 HJB FOR THE ONE ASSET CASE

Let $R^e = R - R^f$ an asset’s gross excess return above the gross risk-free rate. Then 4.4 can be solved for the expected gross excess return by subtracting $R^f_{t+1}$ on both sides:

$$E_t(R^e_{t+1}) = -R^f_{t+1} \cdot Cov_t(m_{t+1}, R_{t+1})$$

Furthermore, employing the inverse relation between the expected value of the SDF and the gross risk-free rate (4.2) one obtains:

$$E_t(R^e_{t+1}) = -\frac{Cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})} = -\frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1}) \rho_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})}$$

Where $\sigma_t(m_{t+1})$ denotes the standard deviation of $m_{t+1}$, $\sigma_t(R_{t+1})$ the standard deviation of $R_{t+1}$ and $\rho_t(m_{t+1}, R_{t+1})$ the correlation between $m_{t+1}$ and $R_{t+1}$.

A lower bound on the volatility of the SDF can then be found by taking absolute values on both sides of 4.5:

$$|E_t(R^e_{t+1})| = \left| -\frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1}) \rho_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})} \right| = \left| \rho_t(m_{t+1}, R_{t+1}) \right| \frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1})}{E_t(m_{t+1})},$$

where obviously:

$$|\rho_t(m_{t+1}, R_{t+1})| \frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1})}{E_t(m_{t+1})} \leq \frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1})}{E_t(m_{t+1})},$$

and therefore:

$$|E_t(R^e_{t+1})| \leq \frac{\sigma_t(m_{t+1}) \sigma_t(R_{t+1})}{E_t(m_{t+1})} \Rightarrow \sigma_t(m_{t+1}) \geq E_t(m_{t+1}) \frac{|E_t(R^e_{t+1})|}{\sigma_t(R_{t+1})}$$

Inequality 4.6 represents a lower bound on the volatility of an SDF for the single asset case and when a risk-free rate exists.

10 I make use of the fact that both $\sigma_t(m_{t+1})$ and $\sigma_t(R_{t+1})$ are non-negative as per definition. Furthermore, $E_t(m_{t+1})$ is non-negative as $m_{t+1}$ is non-negative (see Section 2.3).
4.1.4 HJB FOR THE MULTIPLE ASSET CASE

Hansen and Jagannathan (1991) were able to derive a lower bound on the volatility of an SDF for the case of multiple assets and when there is no risk-free asset in the market. They draw on the fact that the de-meaned SDF can be decomposed into a linear combination of available de-meaned asset returns plus an orthogonal component $\varepsilon$:

$$m_{t+1} - E_t(m_{t+1}) = (R_{t+1} - E_t(R_{t+1}))^T b + \varepsilon,$$  \hspace{1cm} 4.7

where $R_{t+1}$ is a vector of time $t+1$ asset gross returns and $T$ denotes a transpose. $b$ is a vector of coefficients from the linear regression of $(R_{t+1} - E_t(R_{t+1}))$ on $(m_{t+1} - E_t(m_{t+1}))$:

$$b = Cov_t(m_{t+1}, R_{t+1}) \cdot Var_t(R_{t+1})^{-1}$$ \hspace{1cm} 4.8

Where $Var_t(R_{t+1})$ is the covariance matrix of asset returns.

The intuition behind 4.7 is that – except in the complete market case – there is a large set of valid SDFs. However, only a unique SDF $m_{t+1}^*$ lies within the market space and hence can be replicated by a linear combination of all available asset returns\footnote{Given a complete market, $m_{t+1}^*$ is the only valid SDF. This is intuitive in the sense that, when the market is complete – i.e. when there are as many assets with linearly independent payoffs as there are states of nature –, then, from an orthogonal perspective, there is no additional direction to move “away” from the market space.}. For $m_{t+1}^*$ a special case of 4.7 holds:

$$m_{t+1}^* - E(m_{t+1}^*) = (R_{t+1} - E(R_{t+1}))^T b$$ \hspace{1cm} 4.9

All other SDFs are positioned outside of and orthogonally to the market space. $m_{t+1}^*$ is the orthogonal projection of these SDFs on the market space. This fact is reflected in the error term $\varepsilon$ in 4.7.

Given relations 4.7 and 4.8, the HJB can be found in the following steps:

Taking the variance of 4.7 and using the fact that $\varepsilon$ is per definition uncorrelated to asset returns yields:

$$Var_t(m_{t+1}) = b^T Var_t(R_{t+1}) b + Var_t(\varepsilon)$$ \hspace{1cm} 4.10

Putting 4.8 into 4.10 and drawing on the fact that $Var_t(\varepsilon)$ is non-negative one obtains:

$$Var_t(m_{t+1}) = Cov_t(m_{t+1}, R_{t+1})^T \cdot Var_t(R_{t+1})^{-1} \cdot Cov_t(m_{t+1}, R_{t+1}) + Var_t(\varepsilon)$$

$$Var_t(m_{t+1}) \geq Cov_t(m_{t+1}, R_{t+1})^T \cdot Var_t(R_{t+1})^{-1} \cdot Cov_t(m_{t+1}, R_{t+1})$$ \hspace{1cm} 4.11
Where:

\[
\text{Cov}_t(m_{t+1}, R_{t+1}) = E_t(m_{t+1}R_{t+1}) - E_t(m_{t+1})E_t(R_{t+1})
\]

Employing that \(E_t(m_{t+1}R_{t+1})\) is the price-vector of gross returns and that gross returns have a price of one (4.1), the relation above can be restated:

\[
\text{Cov}_t(m_{t+1}, R_{t+1}) = t - E_t(m_{t+1})E_t(R_{t+1}), \tag{4.12}
\]

where \(t\) is a vector of ones.

Substituting 4.12 into 4.11 leads to the common representation of the HJB in the multiple asset case:

\[
\text{Var}_t(m_{t+1}) \geq (t - E_t(m_{t+1})E_t(R_{t+1}))^T \times \text{Var}_t(R_{t+1})^{-1} \times (t - E_t(m_{t+1})E_t(R_{t+1})), \tag{4.13}
\]

4.2 Derivation of the Equity Premium Puzzle

In order to evaluate \((\mu(m), \sigma^2(m))\)-pairs according to relation 4.13, it is necessary to specify the consumption process. This will be done in the next section. Based on this specification, explicit formulas for mean and variance of the SDF are introduced in Section 4.2.2. Finally, I explicate the calculation of implied risk aversion for both economies introduced so far in order to show that the historically observed equity premium cannot be explained by these models (Section 4.2.3).

4.2.1 Specification of the Consumption Process

Tallarini (2000, p. 511) considers two different specifications of the log consumption process in order to investigate the Equity Premium Puzzle: 1. a random walk model and 2. a trend stationary model:

1. Random walk model:

\[
\log(c_t) = \log(c_0) + t\mu + \sum_{i=1}^{t} \varepsilon_i, \ \varepsilon_i \ i.i.d. N \sim (0, \sigma^2_\varepsilon) \tag{4.14}
\]

2. Trend stationary model:

\[
\log(c_t) = \rho^t \log(c_0) + \mu t + (1 - \rho^t)\xi + \sum_{i=1}^{t} (\varepsilon_i \rho^{t-i}), \ \varepsilon_i \ i.i.d. N \sim (0, \sigma^2_\varepsilon) \tag{4.15}
\]

He legitimizes his choice by emphasizing that it is common in the literature to parameterise consumption “(…) in one of two ways: difference stationary [e.g. a random walk model] or trend stationary with highly serially

---

12 See Barillas et al. (2009, p. 2393) for the versions given here.
correlated deviations from trend.” (p. 511). Furthermore, Barillas et al. (2009, p. 2393) assert that Tallarini chooses these specifications, as they explain U.S. post World War II consumption data well.

In the following, I will use the random walk model (4.14) for two reasons: First and most importantly, Barillas et al. show that, surprisingly, both stochastic specifications of the consumption process provide the same results for an investor with ambiguity averse preferences as specified in Section 5. Second, for a cross-country analysis as conducted in this paper, it is more consistent to use the random walk because the same model specification applies to all the countries, whereas a trend stationary model requires accurately specifying and estimating the consumption process for every single economy. This is both analytically as well as empirically more involved, but at the same time it does not provide any potential for further insights. In contrast, it may well pose the risk of misspecification and imprecise results.

Accordingly, the log consumption growth is given by:

\[
\log \left( \frac{c_t}{c_{t-1}} \right) = \log(c_t) - \log(c_{t-1})
\]

\[
= (\log(c_0) + t\mu + \sum_{i=1}^t \varepsilon_i) - (\log(c_0) - (t - 1)\mu - \sum_{i=1}^{t-1} \varepsilon_i)
\]

\[
= \mu + \varepsilon_t
\]

And therefore:

\[
E \left( \log \left( \frac{c_t}{c_{t-1}} \right) \right) = E(\mu + \varepsilon_t) = E(\mu) + E(\varepsilon_t) = \mu \quad 13
\]

\[
Var \left( \log \left( \frac{c_t}{c_{t-1}} \right) \right) = Var(\mu + \varepsilon_t) = Var(\varepsilon_t) = \sigma_{\varepsilon}^2 \quad 14
\]

\[
\Rightarrow \log \left( \frac{c_t}{c_{t-1}} \right) \sim N(\mu, \sigma_{\varepsilon}^2) \quad 4.17
\]

Hence, the random walk model as specified in 4.14 implies an extremely simple log-normal consumption growth with mean \( \mu \) and variance \( \sigma_{\varepsilon}^2 \).

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13 Per definition of \( \varepsilon_t \) it holds that \( E(\varepsilon_t) = 0 \).
14 ...and that \( Var(\varepsilon_t) = \sigma_{\varepsilon}^2 \).
4.2.2 Moments of the SDF

In order to be able to find \((\mu(m), \sigma^2(m))\)-loci which comply with the empirical distribution characteristics of consumption growth, it is necessary to derive closed form solutions for mean and variance of the SDF.

4.2.2.1 Economy I

For the standard case with CRRA time-separable intertemporal preferences and power utility the SDF is given by 2.8:

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\gamma} \]

Under log-normal consumption growth (4.17), the following first and second moments can be found for \( m_{t+1} \):

\[ E_t(m_{t+1})^{(y)} = E_t \left( \beta \left( \frac{c_{t+1}}{c_t} \right)^{\gamma} \right) = \beta E_t \left( e^{\log \left( \frac{c_{t+1}}{c_t} \right)^{\gamma}} \right) = \beta E_t \left( e^{\gamma \log \left( \frac{c_{t+1}}{c_t} \right)} \right) \]

\[ \Rightarrow E_t(m_{t+1})^{(y)} = \beta e^{-\gamma \mu + 0.5 \gamma^2 \sigma_t^2} \]

\[ E_t(m_{t+1}^2) = \beta^2 E_t \left( e^{\log \left( \frac{c_{t+1}}{c_t} \right)^{2\gamma}} \right) = \beta^2 E_t \left( e^{2\gamma \log \left( \frac{c_{t+1}}{c_t} \right)} \right) \]

\[ \Rightarrow E_t(m_{t+1}^2)^{(y)} = \beta^2 e^{-2\gamma \mu + 2\gamma^2 \sigma_t^2} \]

It can then be shown that (see Appendix 8.4 for the derivation):

\[ Var_t(m_{t+1}) = \beta^2 e^{-2\gamma \mu + \gamma^2 \sigma_t^2} e^{\gamma^2 \sigma_t^2} (e^{\gamma^2 \sigma_t^2} - 1) \]

4.2.2.2 Economy II

For the case with EZ-preferences, power utility and log-normal consumption growth it is mathematically rather involved to find the analogues to 4.18 and 4.20:

First, it is necessary to find an explicit form of the recursive utility in 3.5e. If the consumption process follows a geometric random walk as in 4.14, according to Barillas et al. (2009, p. 2395) the solution is given by:

\[ V_t = \frac{\beta}{(1-\beta)^2} \left( \mu - \frac{\sigma_t^2}{(1-\gamma)} \right) + \frac{1}{1-\beta} \log \left( \frac{c_t}{c_{t+1}} \right) \]

15 This follows from the characteristics of the moment generating function of random log consumption growth: The (one-period) MGF is defined as \( E(e^{x}) \). For a normally distributed random variable \( X \sim N(\mu, \sigma^2) \) (such as the log consumption growth) the MGF writes: \( E(e^{x}) = e^{\mu + 0.5 \sigma^2} \). In the case above we have a normally distributed random variable \(-\gamma \log \left( \frac{c_{t+1}}{c_t} \right) \sim N(-\gamma \mu, \gamma^2 \sigma_t^2)\), which justifies the solution.
This explicit representation of the investor’s intertemporal utility can be found by means of dynamic programming: $3.5e$ can be interpreted as a Bellman equation and under log-normal consumption growth $4.21$ is the solution to it. Substituting $4.21$ into the formula for the SDF in Economy II (3.6) and solving for the first and second moments yields (Barillas, Hansen, & Sargent, 2009, p. 2396):

$$E_t(m_{t+1})(\gamma) = \beta \exp \left( -\mu + \frac{\sigma^2}{2} (2\gamma - 1) \right)$$  \hspace{1cm} 4.22

$$\text{Var}_t(m_{t+1})(\gamma) = E_t(m_{t+1})^2 * (\exp(\sigma^2 \gamma^2) - 1)$$  \hspace{1cm} 4.23

### 4.2.3 Calculation of implied risk aversion

The minimum risk aversion implied in an equity market can be calculated in two steps based on the results from Sections 4.1.4 and 4.2.2.1 for Economy I, Sections 4.1.4 and 4.2.2.2 for Economy II respectively:

First, the HJB has to be found. The procedure is identical for both economies. If the vector of expected gross returns $E_t(R_{t+1})$ and the covariance matrix of asset returns $\text{Var}_t(R_{t+1})$ are known, $4.13$ represents $\text{Var}_t(m_{t+1})$ as a function of $E_t(m_{t+1})$ alone; as soon as the expected return vector and the covariance matrix of a representative set of assets have been estimated, the HJB can be found by solving $4.13$ for increasing values of $E_t(m_{t+1})$. In order to illustrate the Equity Premium Puzzle for the U.S. stock market, I use the estimates given by Barillas et al. (2009, p. 2391). They use two assets: The value-weighted NYSE portfolio (as a representation of the U.S. stock market) and the return on a three month t-bill (as a proxy for the risk-free rate). They consider quarterly data for the 1948:Q2 – 2006:Q4 period and find an average excess return of $1.95\%$ and a standard deviation of $7.67\%$ for the NYSE portfolio, an average return of $0.32\%$ and a standard deviation of $0.61\%$ for the t-bill, respectively. I assume that the two assets are mutually uncorrelated, as the historical covariance is not reported.

Second, I trace out $(\mu(m), \sigma^2(m))$-pairs for increasing levels of risk aversion, based on historical distribution properties of consumption growth, i.e. mean and variance of log consumption growth. For this purpose, I draw on relations 4.18 and 4.20 for Economy I, 4.22 and 4.23 for Economy II, respectively: For both economies the expected value of the SDF (see 4.18 and 4.22) is a function of the risk aversion parameter $\gamma$ alone, given that the subjective discount factor $\beta$, mean $\mu$ and variance $\sigma_v^2$ of log consumption growth are known. Once again, I use estimates provided in Barillas et al. (2009, p. 2393) as sample moments of log consumption growth: For the random walk model they find a mean $\mu = 0.495\%$ and a standard deviation $\sigma_v = 0.500\%$. Regarding the subjective discount factor I assume $\beta = 0.995$ following both Tallarini (2000, p. 514) and Barillas et al. (2009, p. 2392). After the insertion of these values in the formula for the expected value of the SDF (4.18 for Economy I and 4.22 for Economy II), I solve the corresponding equation for an arbitrary set of increasing risk aversion parameters. Because $E_t(m_{t+1})$ is then known for every element in this set, 4.20 (for Economy I) and 4.23 (for Economy II) can be used to find the variance of the SDF for the same set of risk aversion parameters. This procedure results in a set of $(\mu(m), \sigma^2(m))$-loci, each of which is associated with a specific risk aversion parameter $\gamma$. 

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This explicit representation of the investor’s intertemporal utility can be found by means of dynamic programming: $3.5e$ can be interpreted as a Bellman equation and under log-normal consumption growth $4.21$ is the solution to it. Substituting $4.21$ into the formula for the SDF in Economy II (3.6) and solving for the first and second moments yields (Barillas, Hansen, & Sargent, 2009, p. 2396):

$$E_t(m_{t+1})(\gamma) = \beta \exp \left( -\mu + \frac{\sigma^2}{2} (2\gamma - 1) \right)$$  \hspace{1cm} 4.22

$$\text{Var}_t(m_{t+1})(\gamma) = E_t(m_{t+1})^2 * (\exp(\sigma^2 \gamma^2) - 1)$$  \hspace{1cm} 4.23

### 4.2.3 Calculation of implied risk aversion

The minimum risk aversion implied in an equity market can be calculated in two steps based on the results from Sections 4.1.4 and 4.2.2.1 for Economy I, Sections 4.1.4 and 4.2.2.2 for Economy II respectively:

First, the HJB has to be found. The procedure is identical for both economies. If the vector of expected gross returns $E_t(R_{t+1})$ and the covariance matrix of asset returns $\text{Var}_t(R_{t+1})$ are known, $4.13$ represents $\text{Var}_t(m_{t+1})$ as a function of $E_t(m_{t+1})$ alone; as soon as the expected return vector and the covariance matrix of a representative set of assets have been estimated, the HJB can be found by solving $4.13$ for increasing values of $E_t(m_{t+1})$. In order to illustrate the Equity Premium Puzzle for the U.S. stock market, I use the estimates given by Barillas et al. (2009, p. 2391). They use two assets: The value-weighted NYSE portfolio (as a representation of the U.S. stock market) and the return on a three month t-bill (as a proxy for the risk-free rate). They consider quarterly data for the 1948:Q2 – 2006:Q4 period and find an average excess return of $1.95\%$ and a standard deviation of $7.67\%$ for the NYSE portfolio, an average return of $0.32\%$ and a standard deviation of $0.61\%$ for the t-bill, respectively. I assume that the two assets are mutually uncorrelated, as the historical covariance is not reported.

Second, I trace out $(\mu(m), \sigma^2(m))$-pairs for increasing levels of risk aversion, based on historical distribution properties of consumption growth, i.e. mean and variance of log consumption growth. For this purpose, I draw on relations 4.18 and 4.20 for Economy I, 4.22 and 4.23 for Economy II, respectively: For both economies the expected value of the SDF (see 4.18 and 4.22) is a function of the risk aversion parameter $\gamma$ alone, given that the subjective discount factor $\beta$, mean $\mu$ and variance $\sigma_v^2$ of log consumption growth are known. Once again, I use estimates provided in Barillas et al. (2009, p. 2393) as sample moments of log consumption growth: For the random walk model they find a mean $\mu = 0.495\%$ and a standard deviation $\sigma_v = 0.500\%$. Regarding the subjective discount factor I assume $\beta = 0.995$ following both Tallarini (2000, p. 514) and Barillas et al. (2009, p. 2392). After the insertion of these values in the formula for the expected value of the SDF (4.18 for Economy I and 4.22 for Economy II), I solve the corresponding equation for an arbitrary set of increasing risk aversion parameters. Because $E_t(m_{t+1})$ is then known for every element in this set, 4.20 (for Economy I) and 4.23 (for Economy II) can be used to find the variance of the SDF for the same set of risk aversion parameters. This procedure results in a set of $(\mu(m), \sigma^2(m))$-loci, each of which is associated with a specific risk aversion parameter $\gamma$. 

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Finally, I approximate the implied minimum risk aversion through the use of a trial-and-error algorithm which calculates the \( (\mu(m), \sigma^2(m)) \)-pair for a specific value of \( \gamma \) and evaluates the distance to the HJB. If the distance is above a given tolerance level, the algorithm repeats the calculation (using an adjusted \( \gamma \)) until it finds an SDF which is sufficiently close to the bound. I call this the critical SDF\(^{16}\).

### 4.2.3.1 Economy I

In Figure 1 I plot the HJB and mean / standard deviation pairs of the SDF for \( \gamma = \{1, 10, 20, \ldots, 380, 390\} \). This figure illustrates the Equity Premium Puzzle in its traditional form: As a result of the inverse relationship between RRA/\( \gamma \) and IES discussed earlier, the SDF reaches the HJB far above its vertex: With increasing RRA the standard deviation of IMRS, and accordingly SDF, grows. At the same time, the mean of the SDF declines due to decreasing IES, which moves possible SDFs away from the HJB. Only for extremely high levels of risk aversion this trend is reversed and the SDF starts to approach the HJB. It can be seen that in order to get close to the HJB a relative risk aversion around 390 is required. An associated SDF would exhibit a standard deviation around 6. If these implications are refused, as it is commonly the case in the literature, it is not possible to explain the historical equity premium in Economy I.

![Equity Premium Puzzle in Economy I](image)

**Figure 1. Hansen-Jagannathan Bound and mean / standard deviation pairs of the SDF implied by CRRA time-additive preferences.**

\(^{16}\)A general note: Throughout the thesis I do the calculations within the mean-standard deviation space rather than within the mean-variance space (as implied so far) for the sake of comparability with Tallarini’s (2000) and Barilla et al.’s (2009) results. Accordingly the HJB represents a lower bound on the volatility of an SDF and I use transformed versions of 4.13, 4.20 and 4.23. In the mean-variance space the HJB is a parabola, in the mean-standard deviation space it is a hyperbola. The results for the implied minimum risk aversion obviously remain unaffected by this transformation.
4.2.3.2 Economy II

Tallarini (2000) realised that the trace of the SDF in the \((\mu(m), \sigma(m))\)-space would approach the HJB more directly under recursive preferences because the inverse relation between RRA and IES is dismissed in this case.

Figure 2 shows the HJB and the mean / standard deviation pairs generated by recursive EZ-preferences for \(\gamma = \{1, 5, 10, \ldots, 55, 60\}\). It is obvious that the SDF gets close to the HJB for much lower levels of risk aversion than under CRRA time-separable preferences\(^\text{17}\). When Tallarini found these results, they were clearly a large success and encouraged doubts about the existence of the Equity Premium Puzzle. But still, the estimated levels of risk aversion necessary to explain the Equity Premium Puzzle are considerably too high. And for alternative specifications of the consumption process – e.g. trend stationary models – the implied risk aversion would be even higher.

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\(^{17}\) Tallarini (2000) and also Barillas et al. (2009) only investigate the level of timidity, i.e. risk- or ambiguity aversion, required to find an SDF which lies above the vertex of the HJB. All the results presented in the empirical part of this thesis are based on the true critical SDF, i.e. the first SDF that actually crosses the HJB and not only the level of its vertex. In the particular case considered here, Tallarini’s approach seems reasonable because the HJB is extremely narrow and the results are accordingly highly sensitive to misspecifications of the subjective discount factor. I will revert to this issue in subsequent sections.
5. **Economy III: Hansen-Sargent ambiguity averse preferences**

In their work, Barillas et al. (2009, p. 2389) refer to Lucas’ (2003) critique that risk aversion coefficients around 50 or even higher are far away from what the behaviour of agents in relevant areas – such as the investment, insurance or gambling industry – implies. Lucas argues that it is therefore necessary “(…) to look beyond high estimates of risk aversion (…)” (p. 8) in order to explain the Equity Premium Puzzle. Barillas et al. follow this critique by considering agents’ ambiguity aversion rather than their risk aversion in order to show that the results presented in the preceding sections might be correct, or at least not so far-off reality, if appropriately reinterpreted. For this purpose, they apply ambiguity averse preferences as introduced by Hansen and Sargent (2001) (henceforth HS-preferences) and show that there exist \( \{\mu(m), \sigma^2(m)\} \)-pairs associated with reasonable levels of ambiguity aversion, which comply with the HJB.

In the following, I will shortly introduce the term ambiguity as it is commonly understood in the areas of information economics and decision theory. Before describing the respective economy and the resulting asset pricing implications, I further elaborate on modelling and measuring of ambiguity aversion within a general equilibrium framework.

### 5.1 Ambiguity: an introduction

In decision theory the term ambiguity was first coined by Ellsberg (1961) which questioned Knight’s (1921) notion of risk and uncertainty. This notion, namely the strict separation of risky and uncertain prospects, is still a widely accepted starting point when it comes to modelling decision behaviour of an economy’s agents. Traditionally, risky propositions – i.e. when the probability distribution over the event space is known – are assessed by means of expected utility theory, whereas uncertain propositions – i.e. when the probability distribution over the event space is unknown – are assessed by means of a heuristic decision criterion such as Minimax Loss. These kinds of decision rules under uncertainty imply that the agents are aware of the fact that they act in complete ignorance. Ellsberg shows that there are situations where neither expected utility theory nor heuristic decision rules apply. In such situations decision makers do not know – or at least do not use – the exact probability distribution associated with a relevant future event. At the same time, they are – or assume to be – not completely ignorant in a Knightian sense, either. This decision theoretic anomaly is known as the Ellsberg-Paradox and it exemplarily describes the behaviour of ambiguity averse agents.

Ellsberg (1961, pp. 652-656) illustrates ambiguity averse preferences by employing the following example:

- An urn contains balls of three different colours: red, black and white.
- It is known that there are exactly 30 red balls and 60 black or white balls in this urn. Hence, the proportion of black and white balls is unknown.
- An agent has to choose between lotteries I and II, III and IV, respectively. See Figure 3.
<table>
<thead>
<tr>
<th>Number of balls</th>
<th>30</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lottery / Colour of balls</strong></td>
<td><strong>red</strong></td>
<td><strong>black</strong></td>
</tr>
<tr>
<td>Lottery I</td>
<td>$100</td>
<td>$0</td>
</tr>
<tr>
<td>Lottery II</td>
<td>$0</td>
<td>$100</td>
</tr>
<tr>
<td>Lottery III</td>
<td>$100</td>
<td>$0</td>
</tr>
<tr>
<td>Lottery IV</td>
<td>$0</td>
<td>$100</td>
</tr>
</tbody>
</table>

**Figure 3. Ellsberg-Lotteries.**

Ellsberg conducted this experiment repeatedly with a number of persons and finds that a majority has a preference for I and IV, or in other words, that most people prefer purely risky lotteries to ambiguous lotteries. For an expected utility maximiser exhibiting such preferences, the following inequalities hold:

Based on lotteries I and II:

$$Pr(red) \cdot u($100) > Pr(black) \cdot u($100), \quad \text{where } Pr(red) = \frac{1}{3}.$$  

$$Pr(red) = \frac{1}{3} > Pr(black)$$

Based on lotteries III and IV:

$$Pr(black \cup white) \cdot u($100) > Pr(red \cup white) \cdot u($100), \quad \text{where } Pr(black \cup white) = \frac{2}{3}.$$  

$$Pr(black \cup white) = \frac{2}{3} > Pr(red \cup white)$$

If these results are combined, it follows that:

$$Pr(black \cup white) + Pr(red) = 1$$

$$Pr(red \cup white) + Pr(black) < 1$$

Where $Pr(\cdot)$ is the real probability measure and $u(\cdot)$ a utility function.

These latter relations establish the Ellsberg-Paradox. Note that the events $(black \cup white)$ and $(red)$ are purely risky in the sense that their probabilities are unambiguously known. In contrast, $(red \cup white)$ and $(black)$ are ambiguous events because their probabilities are not definite. Mathematically speaking, in a given probability space $(black \cup white)$ and $(red)$ are elements of the sigma-algebra and therefore measurable, $(red \cup white)$ and $(black)$ are not. As it can be seen above, this circumstance entails that probabilities are non-additive in an ambiguous world. The interpretation of these findings is that ambiguity averse agents perceive favourable (unfavourable) ambiguous events as less (more) likely than if they were purely risky.
The considerations above show that ambiguity averse behaviour is not consistent with the standard expected utility framework which is based on additive probabilities. Furthermore, it is easy to see that Ellsberg’s test persons did not draw on any of the commonly known heuristic decision criteria as they would generally imply indifference between lotteries I and II, III and IV respectively (see Ellsberg (1961, p. 656)) \(^ {18} \). This means that behavioural evidence suggests that ambiguity is a distinct form of uncertainty, additional to risk and Knightian uncertainty.

In praxis, ambiguity can be understood as an information-state “in which information (…) is scanty, marked by gaps, obscure and vague, or on the contrary plentiful and precise but highly contradictory” (Ellsberg, 2001, p. 1).

5.2 Modelling ambiguity aversion

The problem an agent faces in an ambiguous situation as the one described above is that there are one or several relevant events which are not elements of the sigma-algebra, and hence are not measurable. The same is true for the case of Knightian uncertainty. However, under Knightian uncertainty it is commonly assumed that the agent is completely ignorant and therefore draws on a uniform distribution over the state space or alternatively some distribution-free decision criterion in order to determine his actions. In contrast, under ambiguity, an agent does not know the exact probability distribution, either, but he believes to have some information on it. This information may enable him to do a best guess for the probability distribution or to rank different probability distributions according to perceived plausibility. Based on this idea, different approaches for the modelling of ambiguity aversion evolved: The first approach thinks of the agent’s ability to rank different probability distributions according to plausibility as a second-order distribution or a second-order belief. If this second-order distribution is completely specified, the decision problem can be solved (see e.g. Seo (2009) or Klibanoff, Marinacci and Mukerji (2005)). A second approach was introduced by Epstein and Schneider (2008) who use recursive multiple-priors utility (Epstein & Schneider, 2003). They assume that an investor has a family of subjectively acceptable likelihoods in mind and that he maximises his expected utility based on the worst-case conditional probability distribution resulting from this set of likelihoods. A last prominent approach comes from Hansen and Sargent (2001) who use relative entropy – i.e. the “distance” between two models – as a measure of ambiguity. They specify ambiguity averse HS-preferences by means of a robust control problem where a malevolent agent (nature) minimises the investor’s intertemporal utility by choosing the true probability measure and simultaneously a benevolent agent (the investor) maximises his intertemporal utility by choosing optimal consumption. This is the approach employed in Barillas et al. (2009) and accordingly in this thesis. I will introduce the basic concepts in the following section before specifying the respective economy in Section 5.4.

\(^ {18} \) For example, Minimax Loss does not fit the observed preferences because each of the lotteries has the same maximum / minimum.
5.2.1 Ambiguity and Robust Control

The HS-preferences are based on the idea that an agent or decision maker is able and willing to estimate the parameters of a relevant data generating process \( f \). I will follow Hansen and Sargent (2008, pp. 22-23) who call this the approximating model \( f_a \):

\[
y_{t+1} = Ay_t + Bu_t + C\tilde{\varepsilon}_{t+1}
\]

Where \( y_t \) is a state vector, \( u_t \) is a vector of controls available to the decision maker in \( t \) and \( \tilde{\varepsilon}_{t+1} \) is an i.i.d. standard normally distributed random vector of mutually uncorrelated elements. \( A, B \) and \( C \) are matrices which govern the linear transition law in 5.1.

They further assume that an ambiguity averse agent draws on this model as a best guess model but at the same time doubts its specification. In order to account for these doubts, the agent surrounds the approximating model \( f_a \) by a set of subjectively acceptable alternative models \( F_p = \{ f^1_p, f^2_p, \ldots \} \), so called perturbed or distorted models of the form:

\[
y_{t+1} = Ay_t + Bu_t + C(\varepsilon_{t+1} + \omega_{t+1}),
\]

where \( \varepsilon_{t+1} \) is another random vector as specified above and \( \omega_{t+1} \) is a vector process which is determined by the history of the state vector \( y^t = \{ y_t, y_{t-1}, \ldots \} \) through a measurable function \( g_t \):

\[
\omega_{t+1} = g_t(y^t)
\]

Perturbed models are perceived as subjectively acceptable by the agent if they are close to the approximating model in a statistical sense.

In other words, the agent has a complete (but doubtful) specification \( f_a \) of the data generating process. Furthermore, he has a set of (unspecified) nearby models \( F_p \) which he thinks the true specification \( f \) comes from. Hansen and Sargent postulate that in this case an ambiguity averse agent searches for a dynamic decision rule which is robust to perturbations of his approximating model in the sense that it performs decently under every model in \( F_p \). The problem can be formulated as a dynamic maxmin decision rule, a so called robust control problem, as follows (Hansen & Sargent, 2008, pp. 32-33):
Let \( r(y_t, u_t) = -(y^T Q y + u^T R u) \) the agent’s one-period quadratic loss function, where \( Q \) and \( R \) are symmetric matrices which satisfy some assumptions which I will not discuss here. Then:

\[
\begin{align*}
\max_{\{u_t\}_{t=0}^{\infty}} & \min_{\{\omega_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t r(y_t, u_t) \\
\text{s.t.} & \\
& y_{t+1} = A y_t + B u_t + C (e_{t+1} + \omega_{t+1}) \\
& R(\omega) \leq \eta
\end{align*}
\]

Where again \( \beta \) is a subjective discount factor, \( E_\delta \) is the expectation under the perturbed measure and conditioned on time \( t \) information, \( \eta \) is a constant and \( R(\cdot) \) is a measure of model-misspecification.

The objective states that an infinitely-lived agent tries to maximise\(^{19}\) his discounted (worst case) expected lifetime loss by choosing from the set of controls in every period. The agent assumes that simultaneously a malevolent opponent – i.e. nature – tries to harm him by choosing worst case perturbations to his approximating model. The first constraint shows that the evolution of the state process is governed by the perturbed transition law \( 5.2 \). Furthermore, the problem is restricted by a second constraint \( R(\omega) \leq \eta \). This inequality restricts the set \( F_p \) of subjectively acceptable alternative models: \( \eta \) is an upper bound on the amount of model-misspecification (relative to the approximating model \( f_a \)) which an ambiguity averse agent seeks robustness against. Hansen and Sargent (2008, p. 31) use the following representation for \( R(\omega) \):

\[
R(\omega) = 2 E_0 \sum_{t=0}^{\infty} \beta^{t+1} I(\omega_{t+1})
\]

\[
I(\omega_{t+1}) = 0.5 \omega_{t+1}^T \omega_{t+1}
\]

Where \( I(\omega_{t+1}) \) is a measure of the distance between a reference model – in this case the approximating model – and an alternative model associated with the perturbation \( \omega_{t+1} \). \( I(\cdot) \) is called relative entropy which is generally defined as an expected log-likelihood ratio:

\[
I_t(f_p) = E_t \left( \log \left( \frac{f_p(y_{t+1}|y_t)}{f_a(y_{t+1}|y_t)} \right) \right)
\]

Where \( f_p(y_{t+1}|y_t) \) is the conditional density under the perturbed probability measure, \( f_a(y_{t+1}|y_t) \) is the conditional density under the approximating measure and \( E_t \) is the expectation under the perturbed measure.

Against this background, \( \eta \) is intuitively related to the agent’s ambiguity aversion: The larger \( \eta \), the larger and more diverse the set \( F_p \) of models the agent seeks robustness against and hence, the larger the agent’s ambiguity aversion.

\[\text{Note that loss is a negative measure. Maximisation of loss is therefore in the agent’s interest, whereas minimisation is in the interest of his opponent.}\]
Hansen and Sargent call problems of the form 5.3 constrained robust control problems because the agent’s degree of ambiguity aversion is incorporated into the extremisation through the restriction \( R(\omega) \leq \eta \). Under certain conditions\(^{20}\), 5.3 provides identical solutions as the following problem (Hansen & Sargent, 2008, pp. 32-33):

\[
\max_{[u_t]_{t=0}^{\infty}} \min_{(\omega_{t+1})_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (r(y_t, u_t) + \beta \theta \omega_{t+1}^T \omega_{t+1})
\]

s. t. \( y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + \omega_{t+1}) \)  

Where \( \theta \) is a penalty term on the relative entropy associated with the minimising choice of \( \omega_{t+1} \).\(^{21}\)

In the following, I will base my considerations on problem 5.4 which is known as a multiplier robust control problem. \( \theta \) can be considered as the shadow price on the second constraint of 5.3 and, similar to \( \eta \), as a measure of the agent’s ambiguity aversion: If the agent is highly ambiguity averse, \( \theta \) is small\(^{22}\) and the minimising player can choose strong perturbations relative to the approximating model without being penalised excessively. This will lead to a larger and more diverse set of alternative models for which the agent’s decision rule needs to work appropriately. In contrast, if \( \theta \) is large, the minimising player will frequently choose alternative models which are closer to the approximating model as evaluated by relative entropy, because large perturbations are heavily penalised. When \( \theta \) is infinitely large, 5.4 reduces to a standard optimal control problem as the minimising player will always choose \( \omega_{t+1} = 0 \) in order not to be penalised unlimitedly. In this case, the agent is not at all concerned about model-mis specifications.

### 5.3 Measuring ambiguity aversion

\( \theta^{-1} \) is a more convenient representation of ambiguity aversion than \( \theta \) because it comes from a bounded set: \( \theta^{-1} \in [0, \frac{1}{\beta}] \). However, this parameter still lacks in interpretability. Barillas et al. (2009, pp. 2404-2405) use detection error probabilities in order to overcome this shortcoming and to calibrate \( \theta^{-1} \). They exploit the fact that every \( \theta \) is associated with a set of alternative models \( F_p \) and accordingly with a respective worst case model \( f_{wc}^p \in F_p \). In other words, \( f_{wc}^p \) is the specific model from the set \( F_p \) with the largest distance to the approximating model for a particular value of \( \theta \). In a next step, they introduce the idea of an agent who is willing to learn about the true specification of the data generating model \( f \). For this purpose, the agent assigns unconditional probabilities of 0.5 to both the approximating (\( \Pr(f_a) \)) and the worst case model (\( \Pr(f_{wc}^p) \)), which means that he considers both these models equally likely ex-ante. Subsequently, \( T \) observations of the true data generating process are revealed and the agent has to update his beliefs accordingly. He does so

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\(^{20}\) See Hansen and Sargent (2008, pp. 139-212)

\(^{21}\) Recall that \( I(\omega_{t+1}) = 0.5 \omega_{t+1}^T \omega_{t+1} \) is the relative entropy for the specified problem. Then \( (\omega_{t+1}^T \omega_{t+1}) \) is a rescaled form of relative entropy.

\(^{22}\) Hansen and Sargent (2008, pp. 173-212) show that \( \theta \in (0, +\infty) \) where \( \theta \) is called a break down point. Below the break down point, the malevolent opponent dominates the extremisation in 5.4 and thus can choose the perturbations such that the objective is always \(-\infty\).
through the use of a likelihood ratio test, where $L_f$ denotes the likelihood of an observed sample under a certain model $f$ (Hansen & Sargent, 2008, p. 226):

Whenever the ratio $\frac{L_{fa}}{L_{f_{wc}}^{p}} > 1$, the agent conjectures that the approximating model is the true data generating model and vice versa. This procedure allows for two types of errors:

Type I error := $\left( \frac{L_{fa}}{L_{f_{wc}}^{p}} < 1 \right)_{fa}$

Type II error := $\left( \frac{L_{fa}}{L_{f_{wc}}^{p}} > 1 \right)_{f_{wc}}$

A type I error means that the agent opts for the worst case model $f_{wc}^{p}$, although the approximating model $fa$ generates the data. In contrast, a type II error is given if the agent chooses $fa$ when the data truly come from $f_{wc}^{p}$. Barillas et al. (2009, p. 2405) define detection error probabilities as:

$p(\theta^{-1}) := Pr(fa) \times Pr(\text{Type I error}) + Pr(f_{wc}^{p}) \times Pr(\text{Type II error})$ \hspace{1cm} 5.5

Where $Pr(\cdot)$ is the probability operator and, as mentioned above, $Pr(fa) = Pr(f_{wc}^{p}) = 0.5$.

$p(\theta^{-1})$ is a mapping from $[0, \frac{1}{2}]$ to $[0,1]$, i.e. it assigns detection error probabilities to levels of ambiguity aversion. Recall that $f_{wc}^{p}$ is the model in $F_{p}$ with the largest distance to the approximating model $fa$. This implies that $f_{wc}^{p}$ is also the model in $F_{p}$ which is most easily distinguishable from $fa$. Furthermore, note that $p(\theta^{-1})$ provides a measure of how easy it actually is to distinguish $fa$ from $f_{wc}^{p}: p(\theta^{-1})$ is the probability of empirically mistaking $fa$ for $f_{wc}^{p}$ or vice versa. Then, the interpretation of detection error probabilities is that an investor seeks robustness against all alternative models which he can empirically distinguish from his approximating model in no more than $(1 - p(\theta^{-1}) \times 100)$% of cases. Therefore, a low detection error probability implies a high level of ambiguity aversion.

If the specifications of $fa$ and $f_{wc}^{p}$ are known for a particular $\theta$, both models can be simulated $N$ times and the associated detection error probability can be estimated via relation 5.5, where the empirical probabilities of the type I and the type II error are given by:

$Pr(\text{Type I error}) = \frac{\sum_{i=1}^{N} \left( \frac{L_{fa}}{L_{f_{wc}}^{p}} < 1 \right)_{fa}}{N}$ \hspace{1cm} 5.6a

$Pr(\text{Type II error}) = \frac{\sum_{i=1}^{N} \left( \frac{L_{fa}}{L_{f_{wc}}^{p}} > 1 \right)_{f_{wc}}}{N}$ \hspace{1cm} 5.6b
Where \((\frac{|f_{f_d}|}{L_{wc}}) < 1\) \(f_a\), \(\frac{|f_{f_d}|}{L_{wc}} > 1\) \(f_p^{wc}\) respectively, are Boolean expressions.

5.4 Economy III

Barillas et al. (2009, p. 2392) consider a linear state transition law and a consumption plan of the following form:

\[y_{t+1} = Ay_t + C \varepsilon_{t+1}\] 5.7a
\[
\log(c_t) = H y_t \] 5.7b

The transition law 5.7a for this economy is a special case of 5.1 for \(B = 0\), i.e. if the agent’s control, consumption \(c_t\), has no effect on the state transition and optimal consumption is determined by the state evolution \(y^t\) through the consumption policy matrix \(H\).

Hansen and Sargent (2001) give a general definition of ambiguity averse preferences in continuous time. Barillas et al. (2009, p. 2398) draw on these results and use the following adapted discrete time multiplier representation for an infinitely-lived agent’s ambiguity averse time zero preferences:

\[
V_0 = \min_{(g_{t+1})} \sum_{t=0}^{\infty} E \left( \left( \beta^t G_t \left( \log(c_t) + \beta \theta E \left( g_{t+1} \log(g_{t+1}) \left| \varepsilon^t, y_0 \right. \right) \right) \right) \right) \] 5.8a

Where \(E(\cdot)\) is the expectation operator under the measure associated with the approximating model, \(\varepsilon^t\) is the history of \(\varepsilon\) up to time \(t\) and \(G_t\) is a martingale with \(G_{t+1} = g_{t+1}G_t, E(g_{t+1} \left| \varepsilon^t, y_0 \right.) = 1, g_{t+1} \geq 0\) and \(G_0 = 1\).

5.8a shows that Barillas et al. use martingale increments to distort the log consumption process. Let \((\Omega, \mathcal{F}, P_a)\) a probability space and let \(P_a\) the approximating measure on \(\mathcal{F}\) which is implied by the agent’s best guess model \(f_a\). Then \(G\) can be considered as a Radon-Nikodym derivative which transforms \(P_a\) into \(P_{wc}\), the probability measure associated with the worst case alternative model \(f_p^{wc}\):

\[
dP_{wc}(\omega) = G(\omega) dP_a(\omega), \]

where \(\omega\) denotes elements of the state space \(\Omega\).

In other words, the agent in this economy assumes that the malevolent player chooses the true probability measure for the next period indirectly through \(G_{t+1}\).

Let \(E(\cdot)\) denote the expectation operator under \(P_{wc}\), then 5.8a can be reformulated:

\[dP_{wc}(\omega) = G(\omega) dP_a(\omega), \]

\[E(\cdot) = \min_{(g_{t+1})} \sum_{t=0}^{\infty} \]
\[ V_0 = \min_{\{g_{t+1}\}} \sum_{t=0}^{\infty} \mathbb{E}\left( \left( \beta^t (\log(c_t) + \beta \theta E(g_{t+1} \log(g_{t+1}) | \epsilon^t, y_0) | y_0) \right) \right) \] 5.8b

Where \( E(g_{t+1} \log(g_{t+1}) | \epsilon^t, y_0) \) is the one-period conditional relative entropy (Hansen & Sargent, 2008, p. 55).

5.8b illustrates that the life-time utility of an ambiguity averse investor is given by the minimised sum of the (worst case) expected discounted log consumption stream plus the discounted stream of penalties on one-period relative entropy. A rational agent maximises this life-time utility with respect to his control \( c_t \) which turns 5.8a / 5.8b into multiplier robust control problems as introduced in Section 5.2.1.

Barillas et al. (2009, p. 2398) show that problem 5.8a can be represented as a Bellman equation of the form:

\[ G(V(y)) = \min_{\epsilon(\epsilon)} G(\log(c) + \beta \int (g(\epsilon) V(Ay + C\epsilon) + \theta g(\epsilon) \log(g(\epsilon))) \pi(\epsilon) d\epsilon), \] 5.9

Where \( \pi(\cdot) \) is a density function.

The recursive solution to this minimisation problem is given by (Barillas, Hansen, & Sargent, 2009, p. 2399):

\[ V(y) = \log(c) - \beta \theta \log \int e^{\left( \frac{V(Ay + C\epsilon)}{\theta} \right)} \pi(\epsilon) d\epsilon, \] 5.10a

or similarly by:

\[ V_t = \log(c_t) - \beta \theta \log \left( E_t \left( e^{\left( \frac{-V_{t+1}}{\theta} \right)} \right) \right), \] 5.10b

5.10b will be used as the intertemporal preference specification for the representative investor of Economy III.

Note that the representative investor in Economy III exhibits identical preferences as the representative investor in Economy II (see 3.5e), where:

\[ \theta = \frac{1}{(1-\beta)(1-\gamma)} \] 5.11

Hence, type II and type III representative investors are observationally equivalent (Barillas, Hansen, & Sargent, 2009, p. 2399): The level of implied \( \theta \) in a specific equity market will be the same for both Economy II and Economy III. However, the interpretation of results will differ: A type II investor cares about risk whereas a type III investor cares about model-misspecification. Accordingly, in Economy II \( \theta \) (or \( \gamma \)) is a measure of risk aversion, in Economy III it is a measure of ambiguity aversion.

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\[ \text{I use the fact that } E_{\mathbb{P}}(X) = E_{\mathbb{P}}(G(X)), \text{ where } E_{\mathbb{P}} \text{ is the expectation operator evaluated under measure } \mathbb{P} \text{ and } X \text{ is a random variable.} \]
5.5 Asset Pricing Implications

The observational equivalence of the representative investors in Economy II and Economy III implies identical pricing rules for both economies. Therefore, the results from Section 3.2 can be adopted without modification:

The SDF for Economy III is the same as for Economy II (see 3.6). An explicitly ambiguity averse representation for the SDF is then obtained by substituting 5.11 into 3.6:

\[ m_{t+1} = \beta \frac{c_t}{c_{t+1}} \frac{e^{-V_{t+1}}}{E_t \left( e^{-V_{t+1}} \right)} \]  

5.12

Where:

\[ g_{t+1} = \frac{e^{-V_{t+1}}}{E_t \left( e^{-V_{t+1}} \right)} \]  

5.13

In this case, the martingale increment \( g_{t+1} \) can be interpreted as the one-step Radon-Nikodym derivative from 5.8a which transforms the approximating model \( f_a \) into the worst case model \( f_{WA}^{WC} \). Accordingly, it is called the minimising martingale increment (Barillas, Hansen, & Sargent, 2009, pp. 2399-2401). The price of an asset under the worst case model can therefore be written as:

\[ \tilde{p}_t = \tilde{E}_t \left( x_{t+1} m_{t+1} \right) \]

Where again \( \tilde{E}_t (\cdot) \) is the expectation operator evaluated under the worst case measure and \( m_{t+1} = \beta \frac{c_t}{c_{t+1}} \), i.e. the SDF for Economy I if \( \gamma = 1 \).

If there is no ambiguity aversion, i.e. if \( \theta = +\infty \), the minimising martingale increment is equal to 1 and accordingly the SDF in 5.12 is equal to the standard SDF \( \beta \frac{c_t}{c_{t+1}} \) without ambiguity aversion HS-preferences collapse into standard expected utility preferences which implies that a representative investor would assume his approximating model to be true and therefore use the standard SDF to price assets. In an economy with HS-preferences the SDF is decomposed into a standard SDF, which serves as a reference for the case of an ambiguity neutral agent, and an adjustment which incorporates ambiguity aversion. Similar to \( \gamma \) in Economy II, \( \theta \) determines the convexity of \( m_{t+1} \) in the \( \{m_{t+1}, V_{t+1}\} \)-space: High (low) ambiguity aversion / low (high) \( \theta \) leads to a large (small) discount on positive intertemporal utility shocks.

\[ ^{25} \text{Note that this is similar to the case when } \gamma = 1 \text{ in Economy II.} \]
5.6 The Equity Premium Puzzle revisited

In this section, I will re-evaluate the Equity Premium Puzzle under the assumption of an ambiguity averse investor as specified above. In order to do so, I follow the structure of Section 4.2: In the next section the consumption process is specified both for the approximating model $f_a$ and additionally, as required by the ambiguity aversion framework, for the worst case model $f_{wc}^p$. Subsequently, I derive the closed form solutions for mean and variance of the SDF in Economy III. In a last step, I use these results to estimate implied detection error probabilities in the U.S. equity market for the period 1948Q2–2006Q4. The results are nearly identical to those in Barillas et al. (2009, pp. 2404-2407) and show that historically observed equity premia are possibly explained by the investors’ ambiguity aversion.

5.6.1 Specification of the consumption process

I assume that the representative investor in Economy III uses the same reference model as the representative investors in Economy I and II. Namely, the type III investor assumes that consumption follows a geometric random walk and his approximating model is given by model 4.14. Hence, log consumption growth is normally distributed with mean $\mu$ and variance $\sigma^2_e$ (see 4.17) and the error term of the log consumption process is i.i.d. normally distributed with mean 0 and variance $\sigma^2_e$.

Additionally, an ambiguity averse agent is concerned about distorted versions of 4.14. Recall that in this thesis ambiguity is defined as model-misspecification and measured by the statistical distance between an approximating and an alternative model. Furthermore, ambiguity aversion is the fear of model-misspecification and it is represented by the size of the set of alternative models for which the agent seeks a robust decision rule. $\theta$ or alternatively a function of $\theta$, such as the detection error probability $p(\theta^{-1})$ – is a measure of this size and it is completely determined by the worst case model $f_{wc}^p$. For the purpose of this thesis, it is therefore sufficient to find the correct specification of the worst case model $f_{wc}^p$ additional to the specification of the approximating model $f_a$ which was introduced above.

The general representation of alternative geometric random walk models is (compare 4.14 and 5.2):

$$\log(c_t) = \log(c_0) + t\mu + \sum_{i=1}^t (\varepsilon_i + \omega(\theta)),$$

$\varepsilon_i$ i.i.d. $N \sim (0, \sigma^2_e)$  \hspace{1cm} 5.14

Where $\omega(.)$ is under the control of the malevolent agent and constitutes a distortion to the error term of the approximating model.

Barillas et al. (2009, p. 2402) employ 5.13 to find the minimising martingale for the geometric random walk model. They use this result to transform the approximating measure to the worst case measure and find that under the worst case model, log consumption growth is still normally distributed with mean $\left(\mu - \frac{\sigma^2_e}{(1-\mu)}\right)$ and variance $\sigma^2_e$. Accordingly, the error term of the log consumption process is i.i.d. normally distributed with mean $-\frac{\sigma^2_e}{(1-\mu)}$ and variance $\sigma^2_e$. The specification of the worst case consumption process is therefore given by:
log(c_t) = log(c_0) + t\mu + \sum_{i=1}^{t} \tilde{\epsilon}_i, \quad \tilde{\epsilon}_i \text{ i.i.d. } N \sim \left( -\frac{\sigma^2_{\epsilon}}{(1-\beta)\theta}, \sigma^2_{\epsilon} \right)  

Furthermore, from 5.14 it follows that:

$$\omega(\theta) = -\frac{\sigma^2_{\epsilon}}{(1-\beta)\theta}$$  

5.16

Surprisingly, 5.16 is deterministic which means that only the drift of the approximating model 4.14 is distorted. The diffusion term remains unaltered. 5.15 provides an expedient specification of the worst case consumption process which I will use in Section 5.6.3 to estimate detection error probabilities.

### 5.6.2 Moments of the SDF

To derive an interpretable representation for mean and variance of the SDF in Economy III, I use the respective moments of the SDF in Economy II 4.22 / 4.23 and express them as a function of $\theta$ rather than $\gamma$. For this purpose, I consider 5.11 as a function of $\gamma$ and substitute its inverse into 4.22 and 4.23:

$$\theta(\gamma)^{-1} = \gamma(\theta) = \frac{1}{(1-\beta)\theta} + 1$$

$$\Rightarrow E_t(m_{t+1})(\theta) = \beta \exp \left( -\mu + \frac{\sigma^2_{\epsilon}}{2} \left( 2 \left( \frac{1}{(1-\beta)\theta} + 1 \right) - 1 \right) \right)$$  

5.17

$$\Rightarrow Var_t(m_{t+1})(\theta) = E_t(m_{t+1})^2 \ast \left( \exp \left( \frac{\sigma^2_{\epsilon}}{(1-\beta)\theta} \right) - 1 \right)$$  

5.18

This procedure leads to correct results due to the observational equivalence of type II and type III investors (see Section 5.5).

### 5.6.3 Calculation of implied ambiguity aversion

The calculation of implied ambiguity aversion follows the same two steps as the calculation of implied risk aversion described in Section 4.2.3. Additionally, detection error probabilities are computed in a third step.

First, the HJB is evaluated. It is independent of the representative investor’s preference specification and therefore identical for all three economies (see Section 4.2.3 for the calculation).

Second, preference specific $(\mu(m), \sigma^2(m))$-loci are traced out for different levels of ambiguity aversion. The procedure is exactly the same as outlined for the case with risk aversion (see Section 4.2.3) except that I employ relations 5.17 and 5.18 and that I use varying values for $\theta$ rather than $\gamma$. This results in a set of $(\mu(m), \sigma^2(m))$-pairs and a set of associated ambiguity aversion parameters $\theta$.

Third, for every $\theta$ in this set I follow Barillas et al. (2009, p. 2405) and simulate both the approximating (4.14) and the worst case (5.15) log consumption process $N=10,000$ times. Every simulated log consumption path is of
length $T=235$, which corresponds to the period 1948Q2 – 2006Q4. Next, I evaluate the log-likelihood of every path under both the approximating and the worst case model. I use:

\[
\log(L_a(\mu, \sigma^2)) = \log \left( \frac{\sigma^2}{\pi} \right) + \sum_{t=1}^{T-1} \left( \frac{c_t - c_{t-1}}{2\sigma^2} \right)
\]

5.19

\[
\log(L_{wc}(\hat{\mu}, \hat{\sigma}^2)) = \log \left( \frac{\hat{\sigma}^2}{\pi} \right) + \sum_{t=1}^{T-1} \left( \frac{c_t - c_{t-1}}{2\hat{\sigma}^2} \right)
\]

5.20

Where $L_a(\cdot)$ and $L_{wc}(\cdot)$ denote joint likelihoods evaluated under the approximating and the worst case model and $\hat{\mu} = \mu - \frac{\sigma^2}{\gamma - \theta^2}$. 5.19 and 5.20 are standard joint log-likelihood functions for i.i.d. normally distributed samples, in this case simulated realisations of log consumption growth. As mentioned before, it holds that

\[
\log \left( \frac{c_t}{c_{t-1}} \right) \sim N(\mu, \sigma^2)
\]

under the approximating model and

\[
\log \left( \frac{c_t}{c_{t-1}} \right) \sim N \left( \mu - \frac{\sigma^2}{\gamma - \theta^2}, \sigma^2 \right)
\]

under the worst case model, respectively, which justifies the specifications above.

For a specific $\theta$ this calculations result in a 4xN matrix of likelihoods. For every path there is:

1. The likelihood of the sample generated by the approximating model 4.14 evaluated under the approximating model based on 5.19: $(L_a|f_a)$
2. The likelihood of the sample generated by the worst case model 5.15 evaluated under the approximating model based on 5.19: $(L_{wc}|f_a)$
3. The likelihood of the sample generated by the approximating model 4.14 evaluated under the worst case model based on 5.20: $(L_a|f_{wc})$
4. The likelihood of the sample generated by the worst case model 5.15 evaluated under the worst case model based on 5.20: $(L_{wc}|f_{wc})$

For every of the N consumption paths, the likelihood ratio $(L_a|f_a) / (L_{wc}|f_{wc})$ is then used to test for a type I error and $(L_a|f_{wc}) / (L_{wc}|f_{wc})$ for a type II error. Subsequently, the probability of type I and type II errors is calculated by employing the estimators 5.6a and 5.6b, respectively. Finally, 5.5 is used to calculate the detection error probability, where ex-ante the approximating and the worst case model are assumed to be equally likely: $\Pr(f_a) = \Pr(f_{wc}) = 0.5$.

This procedure is repeated for every $\theta$ in the set. The fact that 5.5 provides a mapping from $\theta$ to detection error probabilities means that it is possible to find an associated value $p(\theta^{-1})$ for every feasible $(\mu(m), \sigma^2(m))$-pair.

As in Economies I and II, I estimate the minimum implied ambiguity aversion by means of a trial-and-error algorithm (see Section 4.2.3): As soon as the critical $\theta$ is found, the associated detection error probability $p(\theta^{-1})$ is calculated. I conduct 10 million simulations of the consumption paths in order to find the critical detection error probability.
5.6.3.1 Economy III

Figure 4 shows the HJB and the trace of the SDF in the \( \{\mu(m), \sigma(m)\} \)-space with decreasing detection error probabilities. The mean / standard deviation pairs are generated under ambiguity averse HS-preferences. As under risk averse EZ-preferences, the SDF quickly approaches the HJB with increasing timidity, in this case ambiguity aversion, of investors. However, the level of ambiguity aversion required to resolve the Equity Premium Puzzle is intuitively more reasonable: A detection error probability of 3\% implies that a representative investor seeks robustness against the set of alternative consumption processes which he can distinguish from his approximating model with a probability of 97\% and less. I.e. the worst case alternative model from this set deviates so strongly from the approximating model, that an investor will confuse the approximating with the worst case model only based on 3 out of 100 samples. Barillas et al. (2009, pp. 2405-2407) believe that detection error probabilities between 15\%-20\% are reasonable. But they find that only a detection error probability around 5\% or lower gets the SDF close to the HJB for their sample. Although they perceive this value as unexpectedly low they argue that an investor exhibiting such a level of ambiguity aversion still seems to be less timid than one with a relative risk aversion above 50. Furthermore, the implied ambiguity aversion is identical for the trend stationary specification of the consumption process which indicates that the passably low level of ambiguity aversion is not due to wrong assumptions about the data generating process.

\textbf{Figure 4.} \textit{Hansen-Jagannathan Bound and mean / standard deviation pairs of the SDF implied by ambiguity-sensitive HS-preferences.}
6. **Empirical results**

In this section, results for three different datasets are presented:

The first two sets contain U.S. market data and reach back to 1889 (yearly data), 1947 (quarterly data) respectively. I use these data to calculate minimum implied risk- and ambiguity aversion on a sliding window basis. The third set contains yearly data since 1970 for 18 different countries. I calculate minimum implied risk- and ambiguity aversion for every country over the whole time period available and use the results for a cross-country analysis.

For every time-period and country under consideration, I estimate mean $\mu$ and standard deviation $\sigma_z$ of real log consumption growth, mean return vector $E_t(R_{t+1})$ and covariance matrix $Var_t(R_{t+1})$ of real equity returns and the risk-free rate. I simulate worst-case and approximating consumption paths 10 million times to estimate the detection error probability implied by the critical $(\mu(m),\sigma^2(m))$-pair.

The calculations exactly follow the procedures described in Sections 4.2.3 for risk aversion and 5.6.3 for ambiguity aversion if not stated otherwise.

6.1 **Ambiguity aversion for the U.S. market: 1889 – 2012**

6.1.1 **Data**


The dataset consists of real annual total returns on the U.S. stock market, real annual rates of the Federal Reserve Board’s 6-month commercial paper series (until August 1997) and real annual rates of secondary market 6-month Certificates of Deposit (until December 2012), and U.S. real personal per capita consumption expenditure. I updated and revised the personal consumption expenditure and the population figures as per 31.12.2012 using the same data source as Shiller, the National Income and Product Account (NIPA) database provided by the Bureau of Economic Analysis (BEA) on its homepage (www.bea.gov).

6.1.2 **Results**

I consider a 50-year sliding window which corresponds to a consumption path of length $T=51$. Accordingly, the first estimate of ambiguity aversion is available as per 31.12.1939. Subsequently, there is one estimate per year until 31.12.2012.
FIGURE 5. DETECTION ERROR PROBABILITY AND RELATIVE RISK AVERSION IN THE U.S. OVER TIME WITH CONSTANT SUBJECTIVE DISCOUNT FACTOR.

Figure 5 shows the evolution of detection error probability and relative risk aversion over the respective period if the subjective discount factor $\beta$ is fixed at 0.9826. The figure reveals several noteworthy aspects:

First, the Equity Premium Puzzle is resolved based on plausibly low levels of ambiguity aversion for most of the periods. The lowest detection error probability, and therefore the highest ambiguity aversion, occurs during the period from 31.12.1932 – 31.12.1982. I find a detection error probability of 1.44% for this period. This implies that investors sought robustness against alternative consumption processes which can be distinguished from the approximating model with a probability equal to or less than 98.56%. Admittedly, this is an extremely large set of alternative models. But given that the periods for which I find detection error probabilities below 5% mostly contain the Great Depression, World War II and the First and Second Oil Crises, I think it is imaginable that consumers of this era were on average extremely uncertain about their future consumption and therefore nearly acted as under complete ignorance. For all the periods before 1982 and after 2000 I find detection error probabilities above 5% and frequently above 10%.

Second, implied levels of risk aversion are a lot higher than commonly accepted in the literature. Only for the first six periods I get values below 10, which is often considered as the threshold for permissible values of RRA. This observation supports the conjecture that Tallarini’s (2000) risk-sensitive preference specification is not able to appropriately explain the Equity Premium Puzzle, almost independently of the time period considered.

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26 This is equivalent to setting $\beta = 0.995$ for quarterly data, which is a common calibration.
Last, Figure 5 shows a very long-term upward trend both in the ambiguity- and the risk aversion of investors. This trend peaked in the 80ies and 90ies and subsequently slightly reversed. Regarding the detection error probability, this trend is mainly driven by two noticeable downward spikes: The first strong decrease in implied detection error probability happened during WWII, between 1941 and 1946. A second major decrease in detection error probability can be observed from 1974-1982, the period of the Oil Crises and the Great Inflation.

These observations raise the question what exactly drives ambiguity aversion. It seems plausible that with the onset of an economic crisis or a major war people start worrying about their financial existence and increasingly doubt the reliability of public institutions or the stability of law, all of which might lead to confusion about the future consumption stream and accordingly to elevated levels of ambiguity aversion. A measure that suggests itself as an indicator for the ambiguity in the market is the inflation rate: First, investors probably perceive inflation as an exogenously given random variable that directly distorts their future consumption. Second, high inflation regimes commonly come with or are caused by civil, political or economic turmoil. I.e. high inflation frequently occurs in genuinely ambiguous environments. Figure 6 shows detection error probabilities and the respective average annual inflation rate over time. A linear regression of the depicted inflation time series on detection error probabilities reveals a strong and statistically significant negative relationship (see Table 1). I conclude that inflation – or possibly the drivers of inflation – seems to be a major source of ambiguity aversion.

![DEP and Inflation over time: 50y sliding window](image)

**Figure 6. Detection Error Probability and Average Annual Inflation Rate in the U.S. over Time with Constant Subjective Discount Factor.**
Another interesting fact resulting from this sliding window analysis is that ambiguity aversion may occasionally prevail in environments characterised by a moderate real equity premium: Only 36% of variations in implied ambiguity aversion can be explained by variations in the level of the historical equity premium (see Table 1).

<table>
<thead>
<tr>
<th>Regression outputs</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg. inflation rate / detection error probability (N=74)</td>
<td>0.305 (0.00)</td>
<td>-6.053 (0.00)</td>
<td>0.74</td>
<td>0.55</td>
<td>0.058</td>
</tr>
<tr>
<td>Avg. real equity premium / detection error probability (N=74)</td>
<td>0.318 (0.00)</td>
<td>-3.690 (0.00)</td>
<td>0.60</td>
<td>0.36</td>
<td>0.069</td>
</tr>
</tbody>
</table>

**Table 1. Possible sources of implied ambiguity aversion: Relation between average historical inflation rate / real equity premium and detection error probability.**

Although detection error probabilities and historical equity premia are clearly negatively correlated, one would possibly expect a stronger relationship between these variables: The assumption behind most of the literature on the topic is that it is hard to match general equilibrium models with historical data due to the large difference between equity and risk-free returns. This assumption is analytically straightforward, as a large real excess return on equities shifts the HJB upwards which leads to higher implied ambiguity- or risk aversion ceteris paribus. However, if the subjective discount factor is assumed to be constant over time, changes in the real risk-free rate can upset this effect: An increasing (decreasing) real risk-free rate pushes the HJB to the left (right). If the critical \( \mu(m), \sigma(m) \)-pair lies on the left side of the HJB – as it is normally the case –, a decrease in the real interest rate will decrease the detection error probability (compare \( \mu(m), \sigma(m) \)-pairs for \( \beta = 0.98 \) with both specifications of the HJB in Illustration 1).
_Illustration 1. The Hansen-Jagannathan Bound for Different Levels of the Risk-Free Rate, Different Levels of the Subjective Discount Factor and a Constant Equity Premium: Higher Levels of Ambiguity- or Risk Aversion Are Required to Reach the HJB if the Real Risk-Free Rate Is Low. If Subjective Discount Factor and Risk-Free Rate Are Inversely Related, the Trace of {μ(m), σ(m)}-Pairs Reaches the HJB Close to Its Vertex._

_Figure 7_ shows that this effect is most accentuated for the period from 1969 until 1978: While the equity premium goes down from about 7% per annum to approximately 4% per annum, the detection error probability remains more or less constant over the same period, although one would expect that the implied ambiguity aversion in the market should strongly decrease and detection error probability should strongly increase. In fact, ambiguity aversion does not dissolve due to a strongly decreasing real risk-free rate, which falls from 1.6% to 0.3% over the same period due to a rise in inflation.

It generally holds that co-movements in the real equity return and the risk-free rate horizontally shift the HJB, no matter what entails the co-movement. But changing inflation seems to be the only systematic source of such shifts. Accordingly, strong variations in the inflation rate uncouple detection error probability from the equity premium. If the inflation rate remained constant over time, the correlation between detection error probability and equity premium would be a lot stronger. This observation leaves room for two different interpretations: First, the Equity Premium Puzzle may also be an inflation rate puzzle because the implied timidity in the market strongly depends on inflation. Or second, the subjective discount factor is not constant over time and possibly even inversely related to the real risk-free rate. In this case, the trace of {μ(m), σ(m)}-pairs crosses the HJB close to its vertex and inflation has a smaller effect on implied risk- or ambiguity aversion (see _Illustration 1_).
In Figure 7, I plot both the detection error probability if the subjective discount factor is kept constant over time and the detection error probability if the subjective discount factor is set to match the inverse of the real risk-free rate over the respective sliding window period. In the latter case, I restrict the subjective discount factor to values smaller than unity in order to keep up the idea of an agent’s impatience. It can be seen that the correlation between detection error probability and the real equity premium remains high even in periods of strongly rising inflation rates if the subjective discount factor is inversely related to the gross risk-free rate. This shows that the calibration of the subjective discount factor plays a decisive role when it comes to interpreting implied ambiguity aversion. In case of a constant $\beta$, inflation seems to be a major source of ambiguity aversion, whereas in the case when $\beta$ is inversely related to the real risk-free rate, ambiguity aversion is almost exclusively explained by the level of the real equity premium. A secondary effect coming with this alternative specification of the subjective discount factor is that the implied detection error probabilities are higher for nearly all time periods because ambiguity aversion does not need to explain additional inflation effects: The lowest detection error probability of 4.61% occurs for the 50-year period ending as per 31.12.1998. Overall, I only find three detection error probabilities below 5% for this specification, which would provide even stronger evidence for the resolution of the Equity Premium Puzzle under ambiguity averse preferences.
But why is the impact of inflation on ambiguity aversion so much stronger if $\beta$ is assumed to be a constant? One possible answer to this question is the following: Consider an investor who exhibits a constant subjective discount factor over time. To such an investor a particular certainty equivalent of random future consumption is of equal worth under any circumstances, i.e. independent of the prevailing inflation- or risk-free rate. If in fact the real risk-free rate is materially lower than the inverse of $\beta$, the investor will probably doubt that he can achieve the expected future utility-levels because the true return on investments is lower than he implicitly assumed. Accordingly, he doubts the specification of his approximating consumption process and seeks robustness against a set of bad alternative consumption processes. In contrast, an investor who sets his subjective discount factor equal to the inverse of the real risk-free rate values a certain future consumption higher in a state of increased inflation or a low real risk-free rate. Accordingly, he is more confident that he can actually achieve the capital appreciation necessary to finance his approximating consumption plan.

It is natural to ask why one should assume a constant subjective discount factor then. But the answer to this question is not as obvious as it seems. First, the subjective discount factor generally captures the notion of an investor’s impatience regarding future consumption. From this point of view, it might be legitimate to assume that there are other factors than the real risk-free rate which determine a representative investor’s propensity...
to postpone consumption – e.g. mortality or morbidity risk, the society’s age structure or liquidity constraints\(^\text{27}\). But even if the real risk-free rate and correspondingly inflation are considered as a good indication for the level of impatience in the market, there are good reasons to assume that agents of an economy only sluggishly adapt their subjective discount factors to changes in the real risk-free rate: If it is asserted that inflation has an impact on the trade-off between consumption today and consumption tomorrow, it will be the expected inflation rate over the respective period which an investor draws on when setting his subjective discount factor. Unless the investor has perfect foresight or continuously updates his expectation, he will always lag the evolution of the real risk-free rate. Under such circumstances, an investor’s subjective discount factor will match the inverse of the real risk-free rate well in times of more or less constant inflation rates; the trace of \((\mu(m),\sigma(m))\)-loci will cross the HJB next to the vertex and the largest part of ambiguity aversion is explained by the amount of the historical real equity premium. Alternatively, if inflation rates sharply increase or decrease, the HJB immediately shifts to the right or to the left, whereas the trace of \((\mu(m),\sigma(m))\)-loci remains nearly unaltered first and only slowly follows the direction of the HJB. This effect results in an inflation-induced, temporary rise in ambiguity aversion.

Clearly, these considerations are intellectual games rather than guidance for the correct characterisation of the representative investor’s subjective discount factor. However, they show that the basic question is maybe not whether the subjective discount factor is exactly constant or perfectly inversely related to the real risk-free rate, but whether it is generally “sticky” or not. If it is, there seem to be inflation-induced shocks to the representative investor’s ambiguity aversion. If not, the difference between real equity returns and the real risk-free rate seems to almost exclusively explain implied ambiguity aversion, which justifies the term Equity Premium Puzzle.

A last finding from this dataset is that implied ambiguity aversion seems to have some predictive power with respect to subsequent long-term real equity returns (see Figure 9). Indeed, a material part of future aggregate fluctuations in real equity prices is explained by variations in implied ambiguity aversion (see Table 2). Note that the explanatory power is the highest with respect to subsequent 20-year returns. However, the results for future 10-year, 15-year, 25-year and 30-year returns are similar and highly statistically significant as well.

As before, I set the subjective discount factor to match the inverse of the real risk-free rate. Therefore, these results implicitly reflect the mean-reverting characteristic of real equity returns in this dataset: Periods of low average equity premia or low average real equity returns were likely to be followed by high long-term real equity returns. I will investigate this relationship in the next section using quarterly data and a shorter sliding window in order to get more timely estimates of ambiguity aversion, which is presumably more suitable for forecasting purposes.

\(^{27}\) See e.g. Chao, Szrek and Sousa Pereira (2009) who find that in areas of high morbidity and mortality an individual’s subjective discount factor is related to its health and survival probability.
**Figure 9.** Detection error probabilities and subsequent annualised 20y real equity returns in the U.S. over time with a subjective discount factor equal to the inverse of the gross risk-free rate.

<table>
<thead>
<tr>
<th>Detection error probability / subsequent 10y real equity return (N=64)</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.028 (0.06)</td>
<td>0.255 (0.00)</td>
<td>0.39</td>
<td>0.14</td>
<td>0.054</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Detection error probability / subsequent 15y real equity return (N=59)</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.028 (0.03)</td>
<td>0.255 (0.00)</td>
<td>0.47</td>
<td>0.20</td>
<td>0.042</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Detection error probability / subsequent 20y real equity return (N=54)</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.024 (0.03)</td>
<td>0.247 (0.00)</td>
<td>0.55</td>
<td>0.29</td>
<td>0.031</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Detection error probability / subsequent 25y real equity return (N=49)</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.032 (0.00)</td>
<td>0.177 (0.00)</td>
<td>0.52</td>
<td>0.25</td>
<td>0.023</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Detection error probability / subsequent 30y real equity return (N=44)</th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.046 (0.00)</td>
<td>0.086 (0.00)</td>
<td>0.42</td>
<td>0.16</td>
<td>0.014</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2. Predictive power of implied ambiguity aversion: Relation between detection error probability and subsequent annualised real equity returns.**

6.2.1 Data

I use quarterly data from different sources for the period from 1947|Q1 – 2012|Q4: Nominal equity returns on the U.S. stock market come from Fama/French and can be downloaded from French’s personal page on the website of Dartmouth’s Tuck School (http://mba.tuck.dartmouth.edu). As a risk-free rate proxy, I employ secondary market 3-month t-bill rate data from the FED (www.federalreserve.gov). The nominal equity returns and t-bill rates are deflated using the seasonally adjusted consumer price index for all urban consumers and all items, provided by the FRED (http://research.stlouisfed.org). For consumption I take seasonally adjusted, chained volume estimates of private final consumption expenditure from the OECD StatExtracts-database (http://stats.oecd.org) and I calculate per capita figures using U.S. population data from the Bureau of economic analysis (www.bea.gov).

6.2.2 Results

I consider a 15-year sliding window of quarterly data. Accordingly, I simulate consumption paths of length T=61 for both the approximating and the worst case alternative model. Implied risk- and ambiguity aversion is calculated on a quarterly basis for the periods ending as per 31.03.1962 until 31.12.2012.

\[\text{DEP and RRA over time: 15y sliding window}\]

\[\text{Detection Error Probability}\]

\[\text{Relative Risk Aversion}\]

\[\text{FIGURE 10. DETECTION ERROR PROBABILITY AND RELATIVE RISK AVERSION IN THE U.S. OVER TIME WITH SUBJECTIVE DISCOUNT FACTOR EQUAL TO THE INVERSE OF THE GROSS RISK-FREE RATE.}\]
*Figure 10* shows detection error probabilities and relative risk aversions for the respective sliding window periods if the subjective discount factor is inversely related to the gross risk-free rate.

It can be seen that, not surprisingly, for quarterly data the variations in implied detection error probability and relative risk aversion are stronger than for the yearly dataset considered in the previous section. In general, the levels of implied ambiguity aversion in the market are slightly higher than for yearly data and a 50-year sliding window: I find temporarily very low values during early 1970s and persistently low levels during mid-1960s and around the turn of the millennium. But still detection error probabilities seem to move within acceptable bounds; the lowest detection error probability of 1.55% occurs during the period from 31.03.1958 – 31.03.1973.

In contrast, the implied relative risk aversion is not even close to reasonable levels for any of the periods under consideration: The lowest level of 49.06 is reached as per 31.12.1987. This amount of risk aversion is still at least five times too high in order to reasonably explain the Equity Premium Puzzle.

Interestingly, for quarterly data the detection error probability and the real equity premium are significantly weaker correlated (see *Figure 11*) than it was the case for the yearly dataset in the previous section, although the subjective discount factor is set to match the inverse of the gross real risk-free rate.

![DEP and equity premium over time: 15y sliding window](chart)

**Figure 11. Detection error probability and average real equity premium in the U.S. over time with subjective discount factor equal to the inverse of the gross risk-free rate.**

*Illustration 2* reveals the explanation for this observation: For the quarterly dataset the HJB is extremely narrow over the complete 1962 – 2012 period due to a very low variance of the risk-free rate and an almost inexistent correlation between equity returns and the risk-free rate. This makes detection error probabilities more
sensitive to horizontal shifts of the HJB relative to the trace of \((\mu(m), \sigma(m))\)-pairs and the equity premium’s relative impact on detection error probability diminishes.

**Illustration 2. Hansen-Jagannathan Bound for Quarterly (left panel) and Yearly (right panel) Data as per 31.12.1989.**

One could expect that if real equity returns are to some extent mean reverting or exhibit a negative autocorrelation in the long-term, then implied ambiguity aversion has some predictive power with respect to real equity returns as long as there is a strong correlation between detection error probabilities and the historical equity premium. This is what we observed in the previous section. As mentioned before, for the case of quarterly data the relation between the historical real equity premium and detection error probabilities is considerably less pronounced than for yearly data (see Table 3). But surprisingly, the explanatory power of detection error probabilities with respect to future equity returns is very high: For subsequent 10-year returns the adjusted coefficient of determination amounts to 0.52 which is even significantly higher than for a regression of the average historical equity premium on subsequent 10-year real equity returns. The coefficient of determination from the latter regression amounts to 0.44. Furthermore, implied detection error probabilities also explain a material part of the variations in subsequent 5-year, 15-year and, to a minor degree, 20-year real equity returns. See Table 3 for the different regression results.
FIGURE 12. DETECTION ERROR PROBABILITY, AVERAGE REAL EQUITY PREMIUM AND SUBSEQUENT 10Y AVERAGE REAL EQUITY RETURNS IN THE U.S. OVER TIME WITH SUBJECTIVE DISCOUNT FACTOR EQUAL TO THE INVERSE OF THE GROSS RISK-FREE RATE.

Regression outputs

<table>
<thead>
<tr>
<th></th>
<th>Intercept (p-value)</th>
<th>Regression coeff. (p-value)</th>
<th>Coefficient of correlation</th>
<th>Adj. coeff. of determination</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg. real equity premium / detection error probability (N=204)</td>
<td>0.156 (0.00)</td>
<td>-4.036 (0.00)</td>
<td>0.51</td>
<td>0.25</td>
<td>0.064</td>
</tr>
<tr>
<td>Detection error probability / subsequent 5y real equity return (N=184)</td>
<td>-0.003 (0.12)</td>
<td>0.172 (0.00)</td>
<td>0.63</td>
<td>0.39</td>
<td>0.014</td>
</tr>
<tr>
<td>Detection error probability / subsequent 10y real equity return (N=164)</td>
<td>0.000 (0.88)</td>
<td>0.140 (0.00)</td>
<td>0.72</td>
<td>0.52</td>
<td>0.009</td>
</tr>
<tr>
<td>Detection error probability / subsequent 15y real equity return (N=144)</td>
<td>0.005 (0.00)</td>
<td>0.097 (0.00)</td>
<td>0.67</td>
<td>0.45</td>
<td>0.007</td>
</tr>
<tr>
<td>Detection error probability / subsequent 20y real equity return (N=124)</td>
<td>0.013 (0.00)</td>
<td>0.033 (0.00)</td>
<td>0.32</td>
<td>0.10</td>
<td>0.007</td>
</tr>
</tbody>
</table>

TABLE 3. POSSIBLE SOURCE AND PREDICTIVE POWER OF IMPLIED AMBIGUITY AVERSION: RELATION BETWEEN HISTORICAL REAL EQUITY PREMIUM AND DETECTION ERROR PROBABILITIES; PREDICTIVE POWER OF DETECTION ERROR PROBABILITIES WITH RESPECT TO SUBSEQUENT REAL EQUITY RETURNS.

So far, I implicitly assumed that ambiguity aversion can (strongly) change over time. One possible explanation for the predictive power of detection error probabilities with respect to future long-term returns is that this assumption is wrong and that the true ambiguity aversion in the market is constant or at least considerably
smoother over time than implied by the results in this and possibly the previous section. To see why, assume that the true ambiguity aversion were approximately constant at a level equivalent to a detection error probability of 10%. Furthermore, assume that deviations of the realised real equity return from the true long-term average real equity return (associated with the true level of ambiguity aversion) can be caused by exogenous events, such as wars, economic crises and alike. During the occurrence of such an event, investors are temporarily inappropriately compensated for the ambiguity they take. Correspondingly, the short term implied detection error probability increases and erroneously indicates lower levels of ambiguity aversion. In the aftermath of the event, investors will require above average real equity returns in order to satisfy the long-term return expectations derived from their true long-term ambiguity aversion.

Such an interaction between exogenous shocks (to short-term) real equity returns and long-term return expectations determined by constant ambiguity aversion could possibly explain aggregate fluctuations of equity prices around their fair value. The spread between short-term and long-term implied ambiguity aversion could lend itself as a good signal for future real equity returns in this case.

6.3 Ambiguity Aversion in the Cross-section of Countries

6.3.1 Data

I consider yearly price data from 31.12.1970 – 31.12.2011 for the cross-country analysis. Total return time series on MSCI country indices (USD) serve as proxies for the evolution of the countries’ equity markets and are provided by Datastream. As in the previous section, I use secondary market 3-month t-bill rates from the FED (www.federalreserve.gov) for the risk-free rate. Equity returns and t-bill rates are deflated using implicit price deflators (USD) from the UN’s National Accounts Main Aggregates Database (http://unstats.un.org). Finally, I use household consumption expenditure at constant prices (USD) and population data from the same source.

6.3.2 Results

I calculate detection error probabilities for a set of 18 different countries and for the whole period from 31.12.1970 – 31.12.2011, which corresponds to a consumption path of length T=42. Input-estimates, implied risk aversions and detection error probabilities for all the countries are provided in the table below.
TRI:

by the most suitable for a cross-
remium which excludes these instruments as a risk
Spain, certainly provide a significant credit p
of credit risk. In contrast short term bonds of crisis
First, the dataset presented here is probabl
time
non
equity returns and the risk
probability of 4.7%.
and 10.7% (Switzerland). The U
Figure 13
R I S K  A V E R S I O N
A N D  R E A L  R I S K
R E T U R N
C O N S U M P T I O N  G R O W T H
T
UK
Switzerland
0.0044
0.0059
1.0485
0.9857
0.0751
0.0144
0.0155
10.71%
66.97
UK
0.0096
0.0109
1.0516
1.0013
0.0673
0.0100
0.0030
19.22%
25.96
USA
0.0087
0.0079
1.0556
1.0145
0.0319
0.0005
0.0009
4.71%
67.46

<table>
<thead>
<tr>
<th>Input-estimates</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_c$</td>
<td>$\sigma_c$</td>
</tr>
<tr>
<td>Australia</td>
<td>0.0084</td>
</tr>
<tr>
<td>Austria</td>
<td>0.0090</td>
</tr>
<tr>
<td>Belgium</td>
<td>0.0083</td>
</tr>
<tr>
<td>Canada</td>
<td>0.0085</td>
</tr>
<tr>
<td>Denmark</td>
<td>0.0055</td>
</tr>
<tr>
<td>France</td>
<td>0.0076</td>
</tr>
<tr>
<td>Germany</td>
<td>0.0080</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>0.0194</td>
</tr>
<tr>
<td>Italy</td>
<td>0.0077</td>
</tr>
<tr>
<td>Japan</td>
<td>0.0092</td>
</tr>
<tr>
<td>Netherlands</td>
<td>0.0056</td>
</tr>
<tr>
<td>Norway</td>
<td>0.0105</td>
</tr>
<tr>
<td>Singapore</td>
<td>0.0153</td>
</tr>
<tr>
<td>Spain</td>
<td>0.0081</td>
</tr>
<tr>
<td>Sweden</td>
<td>0.0060</td>
</tr>
<tr>
<td>Switzerland</td>
<td>0.0044</td>
</tr>
<tr>
<td>UK</td>
<td>0.0096</td>
</tr>
<tr>
<td>USA</td>
<td>0.0087</td>
</tr>
</tbody>
</table>

**Table 4.** Input-estimates and results for different countries: $\mu_c$ denotes mean log consumption growth, $\sigma_c$ standard deviation of log consumption growth, $\mu_{Rm}$ mean real equity return (gross), $\mu_{RF}$ mean real risk-free rate (gross), $\sigma_{Rm}^2$ variance of real equity return, $\sigma_{RF}^2$ variance of real risk-free rate, $\sigma_{Rm,RF}$ covariance between real equity return and real risk-free rate, $p(\theta^{-1})$ implied detection error probability and $\gamma$ implied relative risk aversion. The subjective discount factor is set to 0.98 for all countries.

*Figure 13* illustrates that, except for the U.S., implied detection error probabilities lie between 22.5% (France) and 10.7% (Switzerland). The U.S. shows a somewhat higher level of ambiguity aversion with a detection error probability of 4.7%. Compared to the rest of the countries, this is mainly due to a significantly lower variance of equity returns and the risk-free rate which results in a narrower HJB (see Appendix 8.5). The higher variances of non-U.S. data stem to a material degree from the variations in the respective USD exchange rate which enter time-series directly and / or via the USD-denominated price deflator. Although this is clearly an undesired effect, the dataset presented here is probably the most suitable for a cross-country analysis for two reasons: First, although it is likely that non-U.S. investors do not draw on the 3-month t-bill rate as an approximation for the risk-free rate, it is objectively still the best option because it (probably) exhibits only very moderate levels of credit risk. In contrast short term bonds of crisis-shaken and heavily indebted countries, such as Italy or Spain, certainly provide a significant credit premium which excludes these instruments as a risk-free proxy.
Second, it is extremely difficult to find a consistent set of country-specific short-term government bond rates or interbank rates as far back as 1970.

Despite these shortcomings of the dataset, the results provide some evidence that reasonable levels of ambiguity aversion can explain the equity premia for all the countries considered, although the true levels of ambiguity aversion in the non-U.S. countries might be a bit higher than reported here.

With regard to risk aversion a similar picture can be observed as in the preceding sections: The implied levels of relative risk aversion are consistently too high. The lowest value of 20.69 occurs for Hong Kong. The remaining countries exhibit relative risk aversions above 25.

![Figure 13. Detection error probabilities and relative risk aversions for different countries over the period from 31.12.1970 – 31.12.2011 with consistent subjective discount factor of 0.98.](image)

Cross-sectional variations in the level of ambiguity aversion are hard to explain: The average annual inflation rate over the considered period does not have a statistically significant explanatory power with respect to implied detection error probabilities (see Table 5). However, this relationship is possibly impaired by the influence of exchange rate fluctuations: As discussed above, these fluctuations tend to increase the variance of both the non-U.S. real equity returns in USD as well as the real risk-free rates for the respective countries, which widens the HJB and makes detection error probabilities less sensitive to horizontal shifts of the HJB and therefore inflation.

In Table 5 I present results for a number of linear regressions on detection error probabilities. Additional to the inflation rate’s historical mean I heuristically choose the following regressors:

- **Average gross national savings** in percent of the GDP. *Intuition:* High savings may dispel doubts about consumption plans and therefore imply low ambiguity aversion.
- **Average population. Intuition:* A country’s size may have an impact on the level of perceived complexity of life and accordingly on the level of ambiguity aversion.
- **Average real per capita consumption** in USD. *Intuition:* High historical consumption levels may increase the investors’ fear of consumption cutbacks.

- **Average real per capita GDP** in USD. *Intuition:* People from wealthy nations may be used to a certain living standard and are therefore more eager to avoid unfavourable consumption states.

- **Average of imports plus exports** in percent of the GDP in USD. *Intuition:* The sum of imports and exports in relation to the GDP is a measure of an economy’s globalisation. A high level of economic globalisation may increase complexity of life and ambiguity aversion.

- **Average general government final consumption expenditures** in percent of the GDP. *Intuition:* High government consumption expenditures are a characteristic of well-developed welfare states. Citizens of welfare states may be more certain about their future consumption.

- **Average OECD employment protection indicator:** High employment protection means high income security and may accordingly reduce doubts about the future consumption process.

- **Average annual wages, PPP** in USD. *Intuition:* Wealthy people may fear consumption cutbacks more than poor people.

### Table 5. Different regressions on detection error probability: Possible explanations of cross-country variations in implied ambiguity aversion.

The correct interpretation of the results above is not self-evident. Although, most of the explanatory variables provide extremely low statistical significance and adjusted coefficients of determination around zero, employment protection and average wages seem to have some explanatory power: High employment protection indicates low ambiguity aversion and a high actual wage level indicates high ambiguity aversion. But
given the small sample (N=16)\textsuperscript{28} and the still moderate explanatory power of these regressions, I consider it likely that the results are random. Even more so because of the total number of regressions I run and due to the economic intuition many of these regressions share. For example, one would possibly expect that if the average wage level is able to partially explain implied ambiguity aversion, also average real per capita consumption or average real per capita GDP should have some explanatory power with respect to detection error probabilities because all three variables are measures of a nation’s wealth – and the intuition behind all three regressions is that rich people may show a higher loss aversion than poor people. However, Table 5 shows that the results from these regressions are not consistent. The same is true for the regressions of employment protection and average government consumption expenditure on detection error probabilities: Both the extent of employment protection and government consumption expenditure are indicators for the perceived level of social security in a country, but also the results from these regressions are contradictory.

Overall, I therefore conclude that the results above provide some evidence that cross-sectional variations in ambiguity aversion are not caused by structural differences between countries. But clearly this conclusion has to be backed by further research. Another dimension which may have an influence on cross-country discrepancies in ambiguity aversion is the existence of cultural peculiarities. This dimension is however hardly quantifiable and testable.

\textsuperscript{28} Both for Hong Kong and Singapore, these statistics were not available in the OECD StatExtracts-database.
7. Conclusion

I follow Barillas et al. (2009) and calculate minimum implied ambiguity aversion in equity markets, as measured by detection error probabilities. I do these calculations for a set of major economies for the period from 1970 – 2011 (yearly data) and – on a sliding window basis – for the U.S. for the periods from 1889 – 2012 (yearly data, 50-year sliding window) and 1947 – 2012 (quarterly data, 15-year sliding window), respectively.

I find that for all economies and all time periods considered, implied levels of ambiguity aversion are passably low: The minimum implied detection error probability of 1.44%, and hence the highest implied ambiguity aversion, is observed for yearly U.S. data and for the period from 31.12.1932 – 31.12.1982: the era of the Great Depression, World War II and the Oil Crises. In general detection error probabilities below 5% are extremely rare except if 15-year sliding windows of quarterly data are considered; however, I mainly employ this latter approach in order to get more timely estimates of implied ambiguity aversion for forecasting purposes and I doubt that the respective results reflect the level of true ambiguity aversion in the market, because these estimates react too heavily and immediately to changing market conditions. Overall, the empirical evidence provided in this thesis strongly supports the assumption of Barillas et al. (2009) that ambiguity-sensitive HS-preferences are able to explain the Equity Premium Puzzle.

Similarly, I calculate the relative risk aversions implied by recursive EZ-preferences and find that for all economies and nearly all time periods the implied levels of risk aversion are massively higher than what is commonly considered plausible in the literature – often a relative risk aversion coefficient around 10 is considered as the threshold. For the U.S. I mostly observe implied relative risk aversions around 20-60 (yearly dataset) and around 50-300 (quarterly dataset). In the cross-section of countries I find values between 20 and 75. These findings illustrate the widely accepted fact that even sophisticated specifications of intertemporal preferences do not lead to a resolution of the Equity Premium Puzzle based on reasonably low levels of risk aversion.

I show that different calibrations of a representative investor’s time preferences, i.e. his subjective discount factor, imply different conclusions regarding the source of ambiguity aversion: Constant time preferences indicate the existence of inflation-induced shocks to ambiguity aversion, whereas a subjective discount factor which is inversely related to the gross real risk-free rate suggests that ambiguity aversion is almost exclusively explained by the difference between real equity returns and the real risk-free rate (the equity premium). I argue that the truth may lie somewhere in the middle and that the subjective discount factor is possibly “sticky” and only slowly adapts to the real risk-free rate. In this case, strongly rising inflation rates could temporarily increase ambiguity aversion in the market, which would partly turn the Equity Premium Puzzle into an Inflation Puzzle.

Furthermore, I find that implied ambiguity aversion has some predictive power with respect to subsequent long-term real equity returns and that estimates based on the shorter, 15-year sliding window are better able
to explain these future aggregate fluctuations in equity prices. The analysis shows that for both U.S. datasets
low levels of implied ambiguity aversion seem to be followed by high real equity returns in the long-term.
Implicitly, this relation reflects the existence of long-term mean-reversion in real equity prices in both datasets,
because low historical real equity returns generally imply low implied ambiguity aversion. But surprisingly,
implied ambiguity aversion exhibits significantly more explanatory power with respect to subsequent real-
equity returns than the historical equity premium. I conclude that cyclical patterns in real equity returns may
be the result of exogenous shocks, such as wars or economic crises, in combination with constant long-term
ambiguity aversion of market participants: After periods of exogenously induced below average equity returns,
investors require disproportionately high returns on equity investments in order to satisfy their long-term
return expectations associated with their true (as opposed to implied) ambiguity aversion.

Finally, I investigate cross-country variations in implied ambiguity aversion. Although I find two variables,
namely average wage and employment protection, which explain these variations to some degree, I conclude
that structural differences between countries do not seem to be a systematic source of differing levels of
ambiguity aversion due to 1. the total number of regressions conducted to find these (modest) results, 2.
overall bad consistence of the results and 3. the small sample size.

Based on the preceding considerations I suggest two different foci for further research on the topic: First, the
correct specification of the representative investor’s subjective discount factor and particularly its evolution
over time. Second, quality and consistency of cross-country data, and particularly the adequate approximation
of risk-free rates for different countries. If these challenges were successfully mastered, this would allow more
accurate estimates of implied ambiguity aversion and accordingly a more detailed analysis of the variations in
implied ambiguity aversion both over time and particularly across countries.
8. **APPENDIX**

8.1 **PROOF: INVERSE RELATIONSHIP BETWEEN RRA AND IES**

Power utility is defined as: \( u(c) = \frac{c^{1-Y}}{1-Y} \)

We have:

1. \( \frac{u(c_{t+1})}{u(c_t)} = \left( \frac{c_{t+1}}{c_t} \right)^{1-Y} = \frac{c_{t+1}}{c_t} \cdot c^{-Y} = \left( \frac{c_{t+1}}{c_t} \right)^{1-Y} \)

2. \( \frac{du(c)}{dc} = (1-Y)c^{-Y} = c^{-Y} \)

3. \( \frac{d^2u(c)}{dc^2} = -Yc^{-Y-1} \)

Therefore, RRA is given by:

\[
RRA(c) := - \frac{c \left( \frac{d^2u(c)}{dc^2} \right)}{\left( \frac{du(c)}{dc} \right)} = - \frac{c \left( -Yc^{-Y-1} \right)}{c^{-Y}} = Yc^{-Y} = \gamma
\]

And IES is given by:

\[
IES(c_t, c_{t+1}) := - \left( \frac{\left( \frac{c_{t+1}}{c_t} \right)^{1-Y}}{\left( \frac{d}{dc} \right) \left( \frac{\left( \frac{c_{t+1}}{c_t} \right)^{1-Y}}{u(c)} \right)} \right) = \left( \frac{c_{t+1}}{c_t} \right)^{1-Y} - \left( \frac{c_{t+1}}{c_t} \right)^{1-Y} \left( \frac{c_{t+1}}{c_t} \right) \left( \frac{c_{t+1}}{c_t} \right) = \gamma
\]

\[
\Rightarrow IES(c_t, c_{t+1}) = \frac{1}{RRA(c) - 1}
\]
8.2 **Proof: Functional Form of G(·)**

From 3.1b and 3.2a, the functional Form of $G(·)$ can be calculated in two steps:

1. Calculate inverse of $H(x)$:

$$H(x) = y = \frac{x^{1-\gamma}}{1-\gamma}$$

$$\log(y(1-\gamma)) = (1 - \gamma) \log(x)$$

$$\log\left((y(1-\gamma))^{\frac{1}{1-\gamma}}\right) = \log(x)$$

$$\Rightarrow H^{-1}(y) = x = (y(1-\gamma))^{\frac{1}{1-\gamma}}$$

2. Calculate functional form of $G(·)$:

$$G(U_{t+1}) = H^{-1}(E_tH(U_{t+1}))$$

$$G(U_{t+1}) = \left(E_tH(U_{t+1}) * (1 - \gamma)\right)^{\frac{1}{1-\gamma}}$$

$$G(U_{t+1}) = \left(E_t\left(\frac{U_{t+1}^{1-\gamma}}{1-\gamma}\right) * (1 - \gamma)\right)^{\frac{1}{1-\gamma}}$$

$$\Rightarrow G(U_{t+1}) = \left(E_t(U_{t+1}^{1-\gamma})\right)^{\frac{1}{1-\gamma}}$$ 3.4

8.3 **Proof: SDF for Risk-Sensitive Preferences**

Using 2.6 and 3.5e it can be shown that:

$$m_{t+1} = \frac{\partial \hat{v}_{t}}{\partial c_{t+1}^{1-\gamma}}$$

1. \[ \frac{\partial \hat{v}_{t}}{\partial c_{t+1}} = \frac{1}{c_t} \]

2. \[ \frac{\partial \hat{v}_{t}}{\partial c_{t+1}} = \beta \frac{1}{(1-\beta)(1-\gamma)} E_{t}\left(e^{((1-\beta)(1-\gamma)V_{t+1})}\right)^{\frac{1}{1-\gamma}} \left(1 - \beta\right) e^{((1-\beta)(1-\gamma)V_{t+1})} \frac{\partial \hat{v}_{t+1}}{\partial c_{t+1}} \]

   a. \[ \frac{\partial \hat{v}_{t+1}}{\partial c_{t+1}} = \frac{1}{c_{t+1}} \]

   b. \[ \frac{\partial e^{((1-\beta)(1-\gamma)V_{t+1})}}{\partial V_{t+1}} = (1 - \beta)(1 - \gamma) e^{((1-\beta)(1-\gamma)V_{t+1})} \]

$$\Rightarrow \frac{\partial \hat{v}_{t}}{\partial c_{t+1}} = \beta \frac{1}{(1-\beta)(1-\gamma)} E_{t}\left(e^{((1-\beta)(1-\gamma)V_{t+1})}\right)^{\frac{1}{1-\gamma}} \left(1 - \beta\right) e^{((1-\beta)(1-\gamma)V_{t+1})} \frac{1}{c_{t+1}}$$

$$\Rightarrow m_{t+1} = \frac{\beta^{((1-\beta)(1-\gamma)) e^{((1-\beta)(1-\gamma)V_{t+1})}}}{(1-\beta)(1-\gamma) E_{t}(e^{((1-\beta)(1-\gamma)V_{t+1})})^{\frac{1}{1-\gamma}} c_{t+1}} = \beta \frac{c_t}{c_{t+1}} E_{t}(e^{((1-\beta)(1-\gamma)V_{t+1})}) 3.6$$
8.4 Proof: Variance of SDF

4.18 and 4.19 can be substituted into the following relation in order to find the solution for \( \text{Var}_t(m_{t+1}) \):

\[
\text{Var}_t(m_{t+1}) = E_t(m_{t+1}^2) - (E_t(m_{t+1}))^2
\]

\[
\text{Var}_t(m_{t+1}) = \beta^2 e^{-2\gamma \mu + 2\gamma \sigma^2} - (\beta e^{-\gamma \mu + 0.5 \gamma \sigma^2})^2
\]

\[
\text{Var}_t(m_{t+1}) = \beta^2 e^{-2\gamma \mu} e^{2\gamma \sigma^2} - \beta^2 e^{-2\gamma \mu} \gamma^2 \sigma^2
\]

\[
\Rightarrow \text{Var}_t(m_{t+1}) = \beta^2 e^{-2\gamma \mu + \gamma^2 \sigma^2} (\gamma^2 \sigma^2 - 1)
\]

4.20
8.5 **Figures: HJB for different countries**

![Graphs showing HJB for different countries](image-url)
8.6 MATLAB code

8.6.1 Sub-routine 1: Hansen-Jagannathan Bound

% Calculates the Hansen Jagannathan Bound for a given mean return
% and a corresponding covariance matrix.
% hjbound = hjbound(mu_R, cov_R)
% 
% Input variables:
% 
% mu_R          nx1 vector of gross mean returns.
% cov_R         nxn covariance matrix of gross returns.
% 
% Output variables:
% 
% hjbound       mx2 matrix. Mapping from values of mu_M (first column) to
%               values of std_M (second column). Where mu_M is the expected
%               value of the SDF, std_M the standard deviation of the SDF
%               and m defines the number of gridpoints for which the HJB is
%               evaluated.
% 
% Lukas Plachel (lukas@plachel.ch)

function hjbound = hjbound(mu_R, cov_R)

m = 1000000;
i = ones(length(mu_R),m);

mu_M = [0.8:((1.05-0.8)/(m-1)):1.05];
std_M = (sum(((i-kron(mu_M,mu_R))'*cov_R^(-1)).*...
            (i-kron(mu_M,mu_R))',2)).^0.5;

hjbound = [mu_M' std_M];
end

8.6.2 Sub-routine 2A: \((\mu(m),\sigma(m))\)-loci for risk aversion

% Calculates E(m)-Std(m)-pairs for different values of
% risk-aversion gamma.
% 
% [musigmaM gamma] = musigmaM_risk(mu_c, sigma_c, beta, model, gamma)
% 
% Input variables:
% 
% mu_c         scalar. Mean (net) log consumption growth.
% sigma_c      scalar. Standard deviation of log consumption growth.
% beta         scalar. Subjective discount factor.
% model        string. 'CRRA' uses time-separable CRRA preferences. 'RW'
%               uses recursive EZ-preferences.
% gamma        mx1 vector. Explicit values of gamma for which
%               \((E(m),\text{Std}(m))\)-pairs are estimated.
% 
% Output variables:
% 
% musigmaM     mx2 matrix. Mapping from values of E(m) to values of Std(m)
%               for different levels of risk aversion. m is the number of risk
%               aversion coefficients for which the \((E(m),\text{Std}(m))\)-pairs are
%               evaluated.
function [musigmaM gamma] =...
    musigmaM_risk(mu_c, sigma_c, beta, model, gamma)

switch model
    case 'CRRA'
        mu_M = beta*exp(-gamma*mu_c+0.5*gamma.*sigma_c.^2);
        std_M = (beta^2*exp(-2*gamma*mu_c+gamma.*sigma_c.^2).*...
            (exp(gamma.*sigma_c.^2)-1)).^0.5;
    case 'RW'
        mu_M = beta*exp(-mu_c+(sigma_c^2/2)*(2*gamma-1));
        std_M = (mu_M.*(exp(sigma_c^2*gamma.^2)-1).^0.5);
end

musigmaM = [mu_M std_M];
end

8.6.3 SUB-Routine 2B: \( \{\mu(m), \sigma(m)\}\)-LOCi FOR AMBIGUITY AVERSION

% Calculates E(m)-Std(m)-pairs for different values of
% ambiguity-aversion/detection error probabilities.
% % [musigmaM detErrorProb discEntropy] =...
%    musigmaM_amb(mu_c, sigma_c, beta, T, nmbSim, theta)
%    %
%    % Input variables:
%    %
%    % mu_c         scalar. Mean (net) log consumption growth.
%    % sigma_c      scalar. Standard deviation of log consumption growth.
%    % beta         scalar. Subjective discount factor
%    % T            scalar (int). Number of periods (quarters/years)to be
%                   simulated.
%    % nmbSim       scalar (int). Number of simulations.
%    % theta        mx1 vector. Explicit values of theta for which
%                   {E(M),Std(M)}-pairs and detection error probabilities are
%                   estimated.
%    % detErrorCalc boolean. Calculates detection error probability and
%                   discounted entropy if true.
%    %
%    % Output variables:
%    %
%    % musigmaM     mx2 matrix. Mapping from values of E(M) where M is an SDF
%                   to values of Std(M) for different levels of ambiguity
%                   avarion. m is the number of thetas for which the
%                   {E(M),Std(M)}-pairs are evaluated.
%    % detErrorProb mx1 vector of detection error probabilities associated with
%                   the (E(M),Std(M))-pairs from musigmaM.
%    % discEntropy mx1 vector of discounted entropies associated with the
%                   detection error probabilities from detErrorProb.
%    %
%    % Lukas Plachel (lukas@plachel.ch)

function [musigmaM detErrorProb discEntropy] =...
    musigmaM_amb(mu_c, sigma_c, beta, T, nmbSim, theta, detErrorCalc)

mu_M = beta*exp(-mu_c+(sigma_c^2/2)*(2*((1./(1-beta)*theta))+1)-1));
std_M = (mu_M.*(exp(sigma_c^2*((1./((1-beta)*theta)))+1).^2)-1).^0.5;  
muSigmaM = [mu_M std_M];

switch detErrorCalc
  case true
    detError = detectionError(mu_c, sigma_c, beta, T, nmbSim, theta);
    detErrorProb = detError(:,1);
    discEntropy = detError(:,2);
  case false
    detErrorProb = [];
    discEntropy = [];
end
end

8.6.4 Sub-routine 3: Detection error probabilities

% Calculates detection error probabilities and discounted entropy for % different thetas.
% detError = detectionError(mu_c, sigma_c, beta, T, nmbSim, theta)
% Input variables:
% mu_c      scalar. Mean (net) log consumption growth.
% sigma_c   scalar. Standard deviation of log consumption growth.
% beta      scalar. Subjective discount factor
% T         scalar (int). Number of periods (quarters/years) to be % simulated.
% nmbSim    scalar (int). Number of simulations.
% theta     mx1 vector. Explicit values of theta for which % {E(M),Std(M)}-pairs and detection error probabilities are % estimated.
% Output variables:
% detError  mx2 matrix. Mappings from values of inverted thetas (theta) % to 1. values of detection error probabilities (first column) % and 2. values of discounted entropy (second column).
% Lukas Plachel (lukas@plachel.ch)

function detError = detectionError(mu_c, sigma_c, beta, T, nmbSim, theta)

  nmbThetas = length(theta);

  cApprox(1:T,1:nmbSim,1:nmbThetas) = log(100);
  cWorst=cApprox;

  randSN=randn(T-1,nmbSim,nmbThetas);

  incrementApprox = cumsum(repmat(mu_c,[T-1 nmbSim nmbThetas])+...  
                         sigma_c*randSN,1);
  incrementWorst = cumsum(repmat(mu_c,[T-1 nmbSim nmbThetas])+...  
                         sigma_c*(randSN-permute(repmat((sigma_c/...  
                          ((1-beta)*theta'))',[T-1 1 nmbSim]),[1 3 2])),1);
% Simulate log consumption paths for approx. and worst case model
  cApprox(2:end,:,:) = cApprox(2:end,:,:)+incrementApprox;
  cWorst(2:end,:,:) = cWorst(2:end,:,:)+incrementWorst;

% Calculate simulated returns for approx. and worst case model
  r_cApprox(:,:,:)=cApprox(2:end,:,:)-cApprox(1:end-1,:,:);
  r_cWorst=cWorst(2:end,:,:)-cWorst(1:end-1,:,:);

% log-likelihood of simulated returns under approx. and worst case model
  likelihood(1,:,:)=-(T-1)/2*log(sigma_c^2*2*pi)-
                      (sum((r_cApprox-mu_c).^2,1))/(2*sigma_c^2);
  likelihood(2,:,:)=-(T-1)/2*log(sigma_c^2*2*pi)-
                      (sum((r_cWorst-mu_c).^2,1))/(2*sigma_c^2);
  likelihood(3,:,:)=-(T-1)/2*log(sigma_c^2*2*pi)-
                      (sum((r_cWorst-permute(repmat((mu_c-sigma_c^2./
                      (1-beta)*theta')),1 3 nmbSim)'.^2,1))/
                      (2*sigma_c^2);
  likelihood(4,:,:)=-(T-1)/2*log(sigma_c^2*2*pi)-
                      (sum((r_cApprox-permute(repmat((mu_c-sigma_c^2./
                      (1-beta)*theta')),1 3 nmbSim)'.^2,1))/
                      (2*sigma_c^2);

% log-likelihood ratios
  likelihoodRatio(1,:,:)=likelihood(1,:,:)/likelihood(2,:,:);
  likelihoodRatio(2,:,:)=likelihood(3,:,:)/likelihood(4,:,:);

% type I and type II errors
  detection_error = likelihoodRatio <= 1;

% detection error probabilities
  p_detError = sum(detection_error,2)/nmbSim;

% Output
  detError(:,1) = 0.5*(sum(p_detError));
  detError(:,2)= (beta/(2*(1-beta)))*(-sigma_c./((1-beta)*theta')).^2;
end

8.6.5 Sub-routine 4: Critical value search algorithm

% Finds the first \{E(m),Std(m)\}-pair which lies within the HJB
% and returns the associated critical argument (gamma or theta).
% [criticalSigmaM criticalMuM criticalArgument] = ...
% valueSearch(minHJ, muSigmaM, mu_c, sigma_c,...
%  beta, model, mode, T, nmbSim, argument, algo)
% Input variables:
% minHJ            scalar. Minimum standard deviation of an SDF
% required to reach the HJB.
% muSigmaM         mx2 matrix. Mapping from values of E(m) to values
% of Std(m) for different levels of risk- or
% ambiguity aversion. m is the number of gammas or
% thetas for which the \{(E(m),Std(m))\}-pairs are
% evaluated.
% mu_c             scalar. Mean (net) log consumption growth.
% sigma_c          scalar. Standard deviation of log consumption
% growth.
function [criticalSigmaM criticalMuM criticalArgument] = valueSearch(HJ, muSigmaM, mu_c, sigma_c, beta, model, mode, T, nmbSim, argument, algo)

tolerance = 1e-5;
[~, vertexHJ] = min(HJ(:,2));
HJleft=HJ(1:vertexHJ,:);
HJright=HJ(vertexHJ:end,:);

i=0;
while i==0 || max(hjCompareleft(:,i))==0 || max(hjCompareright(:,i))==0
  i=i+1;
  hjCompareleft(:,i) = ...
  all(repmat(muSigmaM(i,:),size(HJleft,1),1)>HJleft,2);
  compRight(:,1:2) = ...
  repmat(muSigmaM(i,:),size(HJright,1),1)>HJright;
  hjCompareright(:,i) = all([-compRight(:,1:2) compRight(:,2)]);% all([-compRight(:,1) compRight(:,2)]),2);
end

% Find one SDF within and one without the HJB
criticalRow = i;

switch algo
  case 'left'
    criticalHJpair(1,1:2) = HJleft(find(hjCompareleft(:,i),1,'first'),:);
  case 'right'
    criticalHJpair(1,1:2) = HJright(find(hjCompareright(:,i),1,'last'),:);
end

criticalMuSigma(1,:,1) = muSigmaM(criticalRow,:);
criticalMuSigma(1,:,2) = muSigmaM(criticalRow-1,:);
distance(1,:) = [sum((criticalMuSigma(1,:,1)-criticalHJpair(1,1:2)).^2)
% Determine an SDF between the two given SDFs which is closer to the HJB % until the distance between SDF and HJB is smaller than the tolerance.
clear hjCompareleft hjCompareright compRight
criticalArgument(1,1) = argument(criticalRow);
criticalArgument(1,2) = argument(criticalRow-1);
i=1;
while distance(end,1) > tolerance && i < 50
    i = i+1;
    criticalArgument(i,1) = ... 
        mean([criticalArgument(end,1) criticalArgument(end,2)]);
    criticalArgument(i,2) = criticalArgument(i-1,2);
arg = criticalArgument(i,1);
    switch mode
        case 'risk'
            criticalVal = musigmaM_risk(mu_c, sigma_c, beta, model, arg);
        case 'amb'
            criticalVal = musigmaM_amb(mu_c, sigma_c, beta, T, nmbSim,...
                arg, false);
    end

    hjCompareleft = all(repmat(criticalVal,size(HJleft,1),1)>HJleft,2);
    compRight(:,1:2) = repmat(criticalVal,size(HJright,1),1)>HJright;
    hjCompareright = all(-[compRight(:,1)-1 compRight(:,2)],2);
    if max(hjCompareleft)==1 && max(hjCompareright)==1
        switch algo
            case 'left'
                criticalHJpair(i,1:2)=... 
                    HJleft(find(hjCompareleft,1,'first'),:);
            case 'right'
                criticalHJpair(i,1:2)=...
                    HJright(find(hjCompareright,1,'last'),:);
        end
        criticalMuSigma(i,:,1) = criticalVal;
        criticalMuSigma(i,:,2) = criticalMuSigma(i-1,:,2);
    else
        criticalHJpair(i,1:2)=criticalHJpair(i-1,1:2);
        criticalMuSigma(i,:,2) = criticalVal;
        criticalMuSigma(i,:,1) = criticalMuSigma(i-1,:,1);
        criticalArgument(i,2) = criticalArgument(i,1);
        criticalArgument(i,1) = criticalArgument(i-1,1);
    end
    distance(i,:) = [sum((criticalMuSigma(i,:,1)-criticalHJpair(i,1:2))... 
        .^2)^0.5 sum((criticalMuSigma(i,:,2)-criticalHJpair(i,1:2))... 
        .^2)^0.5];
end
if i == 50
    disp(['No critical point found on the ' algo ' side of the HJB'])
criticalSigmaM = -999;
criticalMuM = -999;
criticalArgument = -999;
else
    criticalSigmaM = criticalMuSigma(end,2,1);
criticalMuM = criticalMuSigma(end,1,1);
criticalArgument = criticalArgument(end,1);
end
end

8.6.6 Main routine

% Main routine for the estimation of implied risk- or ambiguity aversion.
% output = main(parameters,estimates)
% Input variables:
% % parameters structure. Contains input parameters:
% .beta scalar. Subjective discount factor
% .model string. 'CRRA' uses time-separable CRRA preferences. 'RW' uses risk-sensitive EZ preferences / ambiguity-sensitive HS-preferences.
% .mode string. 'risk' calculates implied risk aversion. 'amb' calculates implied ambiguity aversion.
% .gamma mx1 vector. Explicit values of gamma for which \( \{E(m),\text{Std}(m)\} \)-pairs are estimated.
% .theta mx1 vector. Explicit values of theta for which \( \{E(m),\text{Std}(m)\} \)-pairs and detection error probabilities are estimated.
% .nmbSim scalar (int). Number of simulations.
% .nmbSimCV scalar (int). Number of simulations for the critical value search.
% .mu_M_low scalar. Lower bound for the evaluation of the HJ-Bound.
% .mu_M_high scalar. Upper bound for the evaluation of the HJ-Bound.
% .criticalValueSearch boolean. True, in order to calculate critical value and argument. False otherwise.
% % estimates structure. Contains input estimates:
% .T scalar (int). Number of periods (quarters/years) to be simulated.
% .mu_c scalar. Mean (net) log consumption growth.
% .sigma_c scalar. Standard deviation of log consumption growth.
% .mu_R nx1 vector of gross mean returns.
% .cov_R nxn covariance matrix of gross returns.
% Output variables:
% % output structure. Contains results:
% .HJbound kx2 matrix. Mapping from values of mu_M (first column) to values of std_M (second column). Where mu_M is the expected value of the SDF M, std_M the standard deviation of the SDF M and k defines the number of gridpoints for which the HJ-Bound is evaluated.
% .muSigmaM mx2 matrix. Mapping from values of E(m) to values of Std(M) for different levels of risk- or ambiguity aversion. m is the number of gammas or thetas for which the \( \{E(m),\text{Std}(m)\} \)-pairs are evaluated.
function output = main(parameters, estimates)
   close all
   tic

   m = 100;
   n = round(parameters.nmbSimCV/m);

   % Calculate HJB
   output.HJbound = hjbound(estimates.mu_R, estimates.cov_R);

   % Calculate \{(E(m),STD(m))\}-pairs (output.muSigmaM) and associated arguments.
   switch parameters.mode
      case 'risk'
         [output.muSigmaM output.argument] = musigmaM_risk(estimates.mu_c,...
            estimates.sigma_c, parameters.beta, parameters.model,parameters.gamma);
      case 'amb'
         [output.muSigmaM output.argument output.discEntropy] =...
            musigmaM_amb(estimates.mu_c, estimates.sigma_c, parameters.beta,...
                estimates.T, parameters.nmbSim, parameters.theta, true);
   end

   % Initialise critical value and argument
   output.criticalValue = -999.99;
   output.criticalArgument = -999.99;

   % Find critical value and argument
   switch parameters.criticalValueSearch
      case true
         switch parameters.mode
            case 'risk'
               [output.criticalValue criticalMuM output.criticalArgument] =...
                  valueSearch(output.HJbound, output.muSigmaM, estimates.mu_c,...
                      estimates.sigma_c, parameters.beta, parameters.model,...
                        parameters.mode, estimates.T, m, output.argument,'left');
               if output.criticalValue==-999
                  [output.criticalValue criticalMuM output.criticalArgument] =...
                 valueSearch(output.HJbound, output.muSigmaM, estimates.mu_c,...
                     estimates.sigma_c, parameters.beta, parameters.model,...
                       parameters.mode, estimates.T, m, output.argument,'right');
               end
            case 'amb'
               [output.criticalValue criticalMuM output.criticalTheta] =...
                  valueSearch(output.HJbound, output.muSigmaM, estimates.mu_c,...
                      estimates.sigma_c, parameters.beta, parameters.model,...
                        parameters.mode, estimates.T, m, parameters.theta,'left');
               if output.criticalValue==-999
                  [output.criticalValue criticalMuM output.criticalTheta] =...
               end
         end
      end

end
valueSearch(output.HJbound, output.muSigmaM, estimates.mu_c, ...
estimates.sigma_c, parameters.beta, parameters.model, ...
parameters.mode, estimates.T, m, parameters.theta, 'right');
end
criticalDetError=zeros(n,2);
for i = 1:n
criticalDetError(i,:) = ...
detectionError(estimates.mu_c, estimates.sigma_c, ...
parameters.beta, estimates.T, m, output.criticalTheta);
end
output.criticalArgument = mean(criticalDetError(:,1));
end

% Plotting: standard figure
figure
plot(output.HJbound(:,1), output.HJbound(:,2))
hold on
plot(output.muSigmaM(:,1), output.muSigmaM(:,2), 'rx');
axis([parameters.mu_M_low parameters.mu_M_high 0 0.8])
xlabel('\mu(m)')
ylabel('sigma(m)')
switch parameters.criticalValueSearch
case true
plot(criticalMuM, output.criticalValue, 'go')
legend('Hansen-Jagannathan Bound', [parameters.model ' ...parameters.mode ': '({\mu(m)},{\sigma(m)})-pairs'], ...
'critical {\mu(m)},{\sigma(m)}-pair', 'Location', 'NorthWest')
switch parameters.mode
case 'risk'
text(criticalMuM-0.035, output.criticalValue, ...
['\gamma=' num2str(output.criticalArgument,4)])
case 'amb'
text(criticalMuM-0.06, output.criticalValue, ...
'P(\theta^-1)= ' num2str(output.criticalArgument,4)])
end
case false
legend('Hansen-Jagannathan Bound', [parameters.model ' ...parameters.mode ': '({\mu(m)},{\sigma(m)})-pairs}', 'Location', ...
'NorthWest')
end

% Plotting: additional figures for ambiguity
switch parameters.mode
case 'amb'
figure
plot(parameters.theta.^-1, output.argument)
axis([0 1.6 0 0.5])
xlabel('\theta^{-1}')
ylabel('p(\theta^{-1})')
legend('detection error probability')

figure
plot(output.discEntropy, output.argument)
axis([0 10 0 0.5])
xlabel('discounted entropy(\eta)')
ylabel('p(\eta)')
legend('detection error probability')
end
toc
end
9. References


10. Internet-References

http://www.bea.gov/national/index.htm
(last accessed on 15.05.2013).

http://www.federalreserve.gov/releases/h15/data.htm
(last accessed on 15.05.2013).

Federal Reserve Bank of St.Louis (FRED): FRED® Economic Data.
http://research.stlouisfed.org/fred2/
(last accessed on 15.05.2013).

French, Kenneth R.: Fama/French data library.
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
(last accessed on 15.05.2013).

http://stats.oecd.org/
(last accessed on 15.05.2013)

Shiller, Robert: Long-term stock, bond, interest rate and consumption data.
(last accessed on 15.05.2013).

United Nations (UN): National Accounts Main Aggregates Database.
(last accessed on 15.05.2013).
11. **Declaration of Authorship**

I hereby declare

- that I have written this thesis without any help from others and without the use of documents and aids other than those stated above,

- that I have mentioned all used sources and that I have cited them correctly according to established academic citation rules.

Lukas Plachel